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FUNDAMENTALS OF THE ANALYTICAL MECHANICS OF SHELLS

by N. A. Kil'chevskiy

Izdatel'stvo Akademii Nauk Ukrainskoy SSR,
Kiev, 1963.

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FUNDAMENTALS OF THE ANALYTICAL MECHANICS OF SHELLS */2

By N. A. Kil'chevskiy

The book discusses analytic methods of constructing elastostatic and elastodynamic systems of differential and integral equations of shell theory without requiring the use of additional assumptions on the deformation of the shells, and also methods of solving the systems of integral equations by reducing them to systems of ordinary differential and algebraic equations.

No use is made of the well-known assumptions that constitute the foundation of classical shell theory, but the author starts out from the general principles of elasticity theory and derives more exact differential equations of the shell theory, of higher order than those of the classical theory.

The book is intended for scientists, post-graduate students and technical university students specializing in the theory of elastic shells.

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* Numbers in margin refer to pagination in foreign text.

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PREFACE

The statics and dynamics of thin elastic shells have been comprehensively studied, but the problem of developing accurate and effective methods of calculating shells still retains all of its current interest.

The object of the present work is to study and systemize various new boundary problems of the shell theory and the methods of their solution derived from the general equations of the statics and dynamics of elastic bodies in the three-dimensionally stress-strain state. This investigation permits an indication of effective methods for solving these boundary problems in both linear and nonlinear form. The interrelation between the three-dimensional problems of the elasticity theory and the two-dimensional problems of the shell theory is analytically established, without requiring the use of auxiliary kinematic hypotheses, such as the most familiar of all, namely that of Kirchhoff and Love.

The general method of investigation is reflected in the arrangement of the Chapters and Sections. The basic concepts of the book are linked with the analytical investigations of the problems of shell theory which the author first took up in 1937-1938.

It must not, however, be assumed that this work is a mere recapitulation of the results of twenty years of work. Those results are given only partially here and must be regarded as preparatory stages in the development of new methods of the analytic theory of shells set forth in the main Chapters of this book. In turn, the book focuses the attention of the reader on the status of the investigations now completed by the author, and constitutes another step toward the next stage in the analytic theory of shells. This is why the book should be regarded as a first part of the study on shell theory. The second part will be ready to go to press in or about 1964.

We have included in the book the principles of the analytic theory of shells, confining ourselves to an extended discussion of the new methods and of the resulting general formulations of the boundary problems of the statics and dynamics of shells, illustrated by a limited number of examples. The second part of the book consists of applications of the theory to specific problems. These applications will, in turn, undoubtedly encourage the further development and enrichment of the theory.

The reader is assumed to be acquainted with mathematical analysis at the university level, theoretical mechanics, theory of elasticity, and classical theory of shells.

The special methods of mathematical analysis, particularly the tensor calculus, also play a considerable role in this work. This is because of the fact that the apparatus of modern higher geometry, namely the tensor calculus, is best suited for the construction of the analytical mechanics of shells. We therefore deemed it expedient to make use of this apparatus, which is gradually

penetrating other fields of technical science. In Chapters I and II we give the basic information on tensor analysis and nonlinear elasticity theory, necessary for the understanding of the theory of shells which is developed thereafter.

The reader will note that certain of our new results in the nonlinear theory of elasticity, which are included in Chapter II, are not further mentioned in the later Chapters concerning the theory of shells. These results, however, will be applied in the next part of the book. /10

Although the title of the present book is "Principles of the Analytic Mechanics of Shells", it does not contain an exhaustive treatment of the applications of the mechanics of Lagrange, Ostrogradskiy, Hamilton, Gauss, Jacobi, Hertz, Chaplygin, and others to the problems of shell theory, although the recent works on analytical mechanics do indicate ways of extending these methods to the mechanics of a continuous medium. The development of the mechanics of shells in this direction is unquestionably of interest, but will require a certain amount of time.

Finally, a word on our method of internal reference to formulas to be found elsewhere in the book, in the literature or in other sources.

We have adopted the following method of indicating references. An entry of the form (II, Sect.3) means a reference to the content of Section 3, Chapter II, while the symbol (III, 11.5) means a reference to eq.(5) of Section 11, Chapter III, and so on.

The literature references are divided into two groups. The first includes the principal sources, which are listed at the end of the book. References to sources not included in this Bibliography, are given as footnotes on the pertaining page.

The Bibliography also includes certain works to which the text does not refer but which helped in formulating the ideas expressed in its content.

This Bibliography, of course, makes no claim at completeness, and the division into two groups is quite arbitrary.

The author expresses his thanks to A.S.Vol'mir, A.D.Kovalenko, and G.N.Savin for checking the manuscript and for valuable comments, and also to A.Kh.Konstantinov and G.L.Komissarova of the Institute of Mechanics, UkrSSR Academy of Sciences, for reading the manuscript, to Z.I.Yasinchuk and L.A.Rudneva who participated in its technical preparation, and to S.G.Shpakov who performed some of the computations.

Kiev, September 1960 to January 1962 N.Kil'chevskiy

INTRODUCTION

A solid, bounded by two boundary surfaces and by a contour surface intersecting the boundary surfaces along the contour curves, is called a shell. Between the boundary surfaces lies the basic (or coordinate) surface, whose selection is arbitrary and is based on the conditions of the specific problem. The object of this selection is to simplify the system of equations of the theory of shells.

The length of the segment of the normal to the basic surface included between the boundary surfaces of the shell is called the thickness of the shell, which will hereafter be denoted by 2h. The thickness of a shell may be either constant or variable. The locus of the midpoints of the segments of the normals that define the thickness of the shell will be arbitrarily termed the middle surface of the shell.

A characteristic feature of the shell is the smallness of the ratio 2h:a, where a is a certain parameter determining the dimensions of the shell. For example, for coverings, a is one of the dimensions defining the projection of the covering on a horizontal plane. Sometimes one of the principal radii of curvature of the middle surface is selected as the parameter a. The geometrical characteristics of shells will be discussed in greater detail in Chapter III.

Shells are common elements of various machines and structures, because of the excellent strength characteristics of designs with thin-walled elements of the shell type.

This book was written in a period of intense development of the statics and dynamics of thin shells. Research is being pressed in various directions, and it would be difficult today to specify any one group of works that could with complete justification be called the basis of the theory. For this reason, the contents of this book are to some extent a reflection of the narrow scientific interests of its author. The choice of the problems touched on in the main part of the book (Chapters III-IV) has been determined by the contents of the well-known monographs by S.A.Ambartsumyan, I.A.Birger, V.V.Bolotin, V.Z.Vlasov, A.S.Vol'mir, A.L.Gol'denveyzer, Kh.M.Mushtar' and K.Z.Galimov, /12 A.I.Lur'ye, V.V.Novozhilov, and O.D.Oniashvili. We have attempted here to consider and analyze those trends of research into shell theory that have not been sufficiently covered by the above-mentioned major works. It must be emphasized that complete attainment of this aim would take us beyond the scope of the present volume, and we have thus had to confine ourselves to the construction of separate fragments of theory which, in our opinion, are a basis for further investigations.

We have thought it expedient to concentrate attention on the development of analytical methods of investigation based, more particularly, on the theory of invariants of coordinate transformations and the analytic definition of the basic geometric operations performed on vector and tensor fields that define

the physical state of a shell. The working apparatus connected with the theory of invariants is the calculus of tensors, together with the principles and various propositions of classical analytical dynamics. We have often made use of a more popular method of approximate representation of functions, the method of constructing approximation functions that satisfy the requirement of the least-square deviation from the approximation function within the region of the approximate representation required.

This method of investigation, in our opinion, also belongs to the analytical mechanics of shells. Here we depart from the classical concept of the field of analytic mechanics, but this formal deviation is thoroughly justified by the essential nature of the method, the more so that it permits to obtain the general equations of motion of a shell and is linked to one of the fundamental variational principles of mechanics, namely the Gauss principle.

To facilitate the reading of the book, the main content is preceded by Chapters I and II, which give in outline the elements of tensor analysis and differential geometry, indicate the elementary geometric properties of shells, and present the relations of the linear and nonlinear theory of elasticity to which later reference will be made. Some of these relations, as mentioned in the Preface, will be used in the next Volume of this work.

Most of the subject matter in the main Chapters (III-V) is connected with the theory of small displacements and deformations of shells, described by linear differential and integro-differential equations. In a number of cases, we consider problems of the nonlinear theory.

The problems of shell dynamics occupy the central position in the book. In Chapter III we consider the little-investigated methods of reducing the three-dimensional problems of the dynamics of homogeneous and inhomogeneous (layered) shells to two-dimensional problems. We analyze the boundary conditions and initial conditions. We compare the various methods of setting up the equations for defining the two-dimensional problems of shell theory. Here /13
we make no use of the simplifying assumptions inherent in the Kirchhoff-Love hypotheses.

The refinements of the equations of the theory of shells, once considered by some scientists to be of purely theoretical interest, have now assumed profound significance in connection with the study of dynamic processes rapidly proceeding in time. Quite meaningful in these cases is the investigation of high-frequency oscillations, which are usually damped more rapidly than those of low frequency.

In studying the brief dynamic processes caused by the short-time action of forces, the effect of the dissipation of energy is not of decisive importance, and we must study a broader segment of the frequency spectrum than in studying slowly proceeding processes*.

* The concept of the speed of a process is of course relative. The natural measure of time in this case, in our opinion, is the time interval required for the spread of dynamic disturbances over the entire region within the shell.

The refined differential equations of the shell theory establish systems of higher order than the system of equations of the classical theory. In this connection, there arises again the problem of the formulation of the boundary and initial conditions completing the formulation of the dynamic boundary problems. These questions are discussed in Chapters III and IV. If we recall the history of the development of modern shell theory, we can clearly apprehend the fundamental difficulties involved in the generalized formulation of the boundary conditions in this field of applied elasticity theory. The author distinctly depicts these difficulties and the controversial nature of a number of propositions advanced by him.

Chapter IV considers various approximate methods of solving the problems of shell mechanics - all methods governed by a single common idea. Their essence resides in the replacement of the shell by an elastic system approximately equivalent to it in respect to certain general features.

In particular, we propose a new approximation method of solving the nonlinear problems of shell theory, closely resembling the method of equivalent linearization known from nonlinear mechanics of systems with one degree of freedom. The perspective of this method is to some extent confirmed by the example for the solution of the problem of stability of a closed cylindrical shell, given in Chapter IV, Sect.6.

We also discuss the construction of a homogeneous shell approximately equivalent in number of layers. This construction is based on the approximation of the Lagrange function of a layered shell by the Lagrange function of a homogeneous shell. To realize this approximation, it was necessary to select a special system of variables which, in the modern literature on mechanics, /14
are termed variable fields.

A preliminary substitution of a homogeneous shell for a layered shell permits, as shown in Chapter IV, to develop a method of approximate determination of fields of displacement, deformation, and stress in layered shells.

Further, the method of constructing the best-square approximations is applicable to the establishment of a new system of equations of motion of a shell element, modifying in this case a general principle of analytical dynamics, namely the D'Alembert-Lagrange principle. Obviously, this involves the connection between the method of least squares and the principle of least constraint, stated by Gauss, which possesses the same degree of generality as the D'Alembert-Lagrange principle. The resultant system of equations has a number of properties permitting its use as a means of solving new problems of shell theory.

The solution of the linear problems of shell theory, and particularly the nonlinear problems, reduces in the general case and at the present level of development to the numerical solution of systems of linear and nonlinear algebraic equations. We have therefore deemed it advisable to consider, in Chapter IV, a method based on the application of interpolation formulas, making it possible to reduce the problem of the motion of shell elements to the solution of a finite system of ordinary differential equations, which are the

Euler-Lagrange equations for the corresponding variational problem. The selection of the variation principle, permitting to set up the equations of motion, depends on the properties of the relations resulting from the boundary conditions of the problem when applying the method of reduction based on the expansion in tensor series of the required components of the displacement vector and the stress tensor. In the general case, these relations are kinematic and belong to types that have not been studied in classic dynamics. In Chapter IV, the question of the formulation of the initial conditions is again investigated.

The mentioned method makes it possible to lay the foundation for a numerical solution of shell theory problems by the use of computers. We term this method the discrete-continuous method, following the terminology proposed by V.Z.Vlasov.

The methods of solution of the boundary problems of shell theory, developed in Chapter V, likewise have the object of laying the foundation for programming the numerical solution of the boundary problems of shell theory. Here we indicate systems of integro-differential equations that result from the theorem of work and reciprocity in both its conventional treatment and as generalized by us to the case of a nonlinearly deformed anisotropic medium. The generalization of the theorem of reciprocal work is presented in Chapter II. In Chapter V a new method is described for reducing the three-dimensional problems of the elasticity theory to two-dimensional problems of the shell theory. This method permits us to describe a field of displacements within /15 a shell by systems of integro-differential equations or, in particular, of integral equations with kernels having peculiar properties, which we have called "focusing" in accordance with the term proposed by K.Lantsosh*.

The integro-differential equations of the shell theory, and the integral equations with focusing kernels, are undoubtedly of considerable theoretical and applied importance. By applying the discrete-continuous method to systems of integro-differential equations with focusing kernels, we approximately reduce the problems of shell dynamics to the solution of relatively simple systems of linear ordinary differential equations or, in more general cases, of nonlinear types. In problems of statics, these systems degenerate to systems of algebraic equations.

Thus, the solutions of the boundary problems of shell statics and dynamics may be found if the coefficients of these ordinary differential equations or the analytical expressions for the kernels with focusing properties are known.

In Chapter V we indicate two methods of constructing these kernels. The first method is based on expansions of the functions, permitting a construction of focusing kernels, in Legendre polynomials. The investigation of these expansions is connected with problems only a step removed from the classical

* Cf.K.Lantsosh, Practical Methods of Applied Analysis. Physical and Mathematical Publishing House, 1961., (Fizmatgiz)

problem of moments*. The first method allows us to find kernels with stronger focusing properties than does the second method. The second method is simpler than the first and does not require complicated additional mathematical investigations, but leads to kernels with weakened focusing properties. In the second volume of this work, we shall indicate specific analytic expressions for focusing kernels for special types of shells and shall present Tables for finding the numerical values for the coefficients of the approximate differential equations of shell dynamics and the algebraic equations of shell statics, that follow from the integro-differential equations with focusing kernels. At the same time, we shall continue the investigation of the analytic properties of the integro-differential equations of shell theory with focusing kernels. These investigations will allow us to find standard programs for the computation of shells of arbitrary form on modern computers.

Such is the plan for future investigations, with the object of establishing a general and effective method for solving problems of shell statics and dynamics, in both linear and nonlinear formulations.

* Cf. N.I. Akhiyezer, The Classical Problem of Moments. Physical and Mathematical Publishing House, 1961.

CHAPTER I /16

ELEMENTS OF TENSOR ANALYSIS AND THEIR APPLICATION
TO THE DIFFERENTIAL GEOMETRY OF SHELLS

Section 1. General Description of the Applications of Tensor Analysis in Shell Theory

Tensor analysis is a modern mathematical apparatus permitting expression, in the most general analytical form, of the fundamental geometric operations performed on the quantities encountered in the investigation of various problems of geometry and physics. Among these operations we may note elementary operations, for instance the measurement of distances between points of space, or measurement of angles between directed segments, and more complex operations, to which reduces the mutual comparison of geometric and physical objects, of a given system of numbers or system of functions of curvilinear coordinates of points in space.

Particularly important is the problem of constructing quantities independent of the choice of the coordinate system. These quantities are termed invariants of coordinate transformations. Tensor quantities are the base for the construction of invariants. The scalars and vectors, which we know from elementary geometry and mechanics, are special cases of tensor quantities.

Most invariants have a definite geometric or physical meaning. Invariants are the basis for the general analytical formulations of the laws of physics, especially those of mechanics. The applications of tensor analysis to the geometry of surfaces are numerous, since here tensor analysis allows us to find expressions of geometric theorems in a simple and yet general form.

The kinematics and kinetics of shells is precisely the branch of mechanics that is internally linked to the geometry of surfaces. The formulation of the boundary problems of shell theory requires the introduction of curvilinear coordinate systems defining the position of the points of the shell.

To set up the kinematic and kinetic equations of shell theory without /17
recourse to the methods of tensor analysis is a cumbersome and complicated operation, and - what is very important - it sometimes involves losses of various terms of the equations. Errors of this kind are most frequently encountered in attempts to set up the equations of shell theory on the basis of elementary "visualized" concepts.

The apparatus of tensor analysis, as already remarked, was developed for the very purpose of solving the problems of geometry and mechanics in curvilinear coordinate systems. This apparatus is most suitable for solving various problems of shell mechanics. All the operations necessary for setting up the kinematic and kinetic equations of shell theory receive, in the framework of

tensor analysis, a rigorous analytic interpretation that makes it unnecessary to appeal to visualized "obvious" ideas.

While tensor analysis does allow us to set up the equations of shell theory, it does not, of course, eliminate the difficulties of solving the corresponding boundary problems. However, it permits us to mark new methods of solving the dynamic boundary problems, based, for instance, on introduction of the functions of kinetic stresses (Bibl.7).

Section 2. Systems of Curvilinear Coordinates. Metrics of Space. The Symbol for Summation.

In passing to a discussion of the mathematical principles of the theory of shells, we assume that the reader is familiar with the rules of operation of vector algebra and the elements of differential geometry.

Shell theory is based on the application of various curvilinear coordinate systems defining the position of the points of the shell. We shall first consider the general properties of an arbitrary coordinate system in three-dimensional space, and shall then illustrate these properties by examples from shell theory.

A system of independent parameters, uniquely determining the position of the points in space, is called a system of curvilinear coordinates. We shall denote them by x^i. In three-dimensional space the number of a coordinate (index) may be 1, 2, 3. On a certain surface, e.g. in two-dimensional space, the index i takes the values 1 and 2.

Let us select a fixed point O in space, and draw the radius vector $\vec{OM} = \vec{r}$ to the point M in space. Since the position of point M is determined by the coordinates x^i ($i = 1,2,3$), the radius vector is a function of x^i:

$$\vec{r}=\vec{r}(x^i). \tag{2.1}$$

We shall assume that $\vec{r}(x^i)$ is a single-valued continuous function, differentiable at least twice with respect to any argument x^i. If two coordinates x^i out of the three are fixed, then eq.(2.1) can be regarded as the equation of a certain curve. This curve is called a coordinate line. Three coordinate lines pass through the point M. Consider the two points $M(x^i)$ and $M'(x^i + dx^i)$. The vector MM' is defined as follows: /18

$$\vec{MM'} = d\vec{r} = \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial x^i} dx^i, \tag{2.2}$$

where Σ is the sign of summation. The derivatives

$$\frac{\partial \vec{r}}{\partial x^i} = \vec{e}_i \qquad (i = 1, 2, 3) \tag{2.3}$$

are vectors directed along tangents to the coordinate lines. The vectors $\vec{e}_i$ form the local coordinate base at the point M. Equation (2.2) defines the expansion of the vector $d\vec{r}$ on the axes of the local coordinate base. The index i in eq.(2.2) is called a dummy index, since it takes no definite value but runs through all values from 1 to 3. Equation (2.2) obviously remains unchanged if the dummy index i is replaced by any other letter. We shall frequently make use, hereafter, of this right to change the dummy indices.

Let us find the distance MM'. We have

$$(MM')^2 = d\vec{r} \cdot d\vec{r} = \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial x^i} dx^i \cdot \sum_{j=1}^{3} \frac{\partial \vec{r}}{\partial x^j} dx^j,$$

or

$$d\vec{r} \cdot d\vec{r} = ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}\, dx^i\, dx^j; \tag{2.4}$$

where $ds = |\vec{MM'}|$; the coefficients g_{ij} are expressed by

$$g_{ij} = g_{ji} = \frac{\partial \vec{r}}{\partial x^i} \cdot \frac{\partial \vec{r}}{\partial x^j} = \vec{e}_i \cdot \vec{e}_j. \tag{2.5}$$

Equation (2.4), defining ds^2, is called the fundamental quadratic form of the quantities dx^i, and the coefficients g_{ij} are called the coefficients of the fundamental quadratic form. Below, we will give a different term for the set of quantities g_{ij}.

The determinant /19

$$\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \mathrm{Det}\,|g_{ik}| = g \tag{2.6a}$$

is called the fundamental determinant.

It follows from eq.(2.5) and the theory of determinants that the fundamental determinant is equal to the square of the volume V of the parallelepiped constructed on the vectors $\vec{e}_i$:

$$V = \pm \sqrt{g}. \qquad (2.6b)$$

Here V is usually taken as a positive quantity. We shall return later to the question of its properties.

The set of quantities g_{ij} permits us to determine the distance between two points. It is easy to show that the set of these quantities allows us to find the angle between the directions of the two vectors $\vec{dr}$ and $\vec{\delta r}$. By making use of expansions of the type of eq.(2.2) of the vectors $\vec{dr}$ and $\vec{\delta r}$ and the properties of a scalar product, we obtain

$$\cos(\vec{dr}, \vec{\delta r}) = \frac{\sum_{i=1}^{3} \sum_{k=1}^{3} g_{ik} dx^i \delta x^k}{ds\, \delta s}. \qquad (2.7)$$

Thus the system of functions of g_{ij} defines the metric of space, e.g. the method of measuring, in a given curvilinear coordinate system, the distances between infinitely near points and the angles between the directions of two vectors.

Equations (2.2), (2.4), and (2.7) may be put into a simpler form by making use of the arbitrary summation convention proposed by A. Einstein.

Hereafter, sums of the form $\sum_{i=1}^{3} a_i b^i$ will conventionally be written simply as $a_i b^i$, omitting the sign $\sum_{i=1}^{3}$. In this case, of course, it will be necessary to indicate it specifically whenever expressions of the form $a_i b^i$ are not sums but monomials. If we use the simplified notation for summation, then, for instance, eq.(2.4) takes the following form:

$$ds^2 = g_{ik}\, dx^i\, dx^k. \qquad (2.8)$$

The abbreviated notation for summation is also applicable to multiple summation.

Another abbreviated notation will be introduced here. We shall denote the operation of differentiation with respect to the coordinate x^i by the symbol ∂_i: /20

$$\partial_i = \frac{\partial}{\partial x^i}; \tag{2.9}$$

Thus,

$$\vec{e}_i = \partial_i \vec{r}. \tag{2.10}$$

Section 3. Metrics in Shells

A curvilinear system of coordinates in a shell involves the preliminary introduction of an undeformed base surface on which a network of coordinate lines x^1 and x^2 is drawn. Most often the base surface is taken to coincide with the middle surface of the undeformed shell. We put

$$\vec{e}_3 = \frac{\vec{e}_1 \times \vec{e}_2}{|\vec{e}_1 \times \vec{e}_2|} = \vec{n},$$

where the $\vec{n}$ are orthonormals to the undeformed basic surface. Thus the normals to the undeformed basic surface of the shell form a system of coordinate lines along which the coordinate x^3 varies. The system of coordinates x^i is the Lagrangian system defining the position of the points of the deformable medium constituting the shell. Under deformations of the shell, the coordinates x^i of a material element of the shell do not vary, but the coordinate lines x^3 deviate from the normals to the deformed basic surface. Let $\vec{r}_0(x^1, x^2)$ be the radius vector of a point of the basic surface of the undeformed shell. Then the radius vector of an arbitrary point of the shell will be expressed by

$$\vec{r}(x^i) = \vec{r}_0(x^1, x^2) + \vec{n}x^3. \tag{3.1}$$

The vectors of the local coordinate base will be expressed by

$$\vec{e}_i = \partial_i \vec{r} + x^3 \partial_i \vec{n} \qquad (i = 1, 2); \tag{3.2a}$$

$$\vec{e}_3 = \vec{n}. \tag{3.2b}$$

If the coordinate lines on the basic surface are taken to coincide with its lines of curvature, as is usually done in shell theory, then, from the Rodrigues formula*, we find /21

$$\partial_i \vec{n} = -k_i \partial_i \vec{r}_0. \tag{3.3}$$

where the k_i denote the principal curvatures of the basic surface:

$$k_i = \frac{1}{R_i}, \tag{3.4}$$

and the R_i are the principal radii of curvature. Thus,

$$e_i = (1 - k_i x^3)\,\partial_i r_0 \qquad (i = 1,\ 2). \tag{3.5}$$

Hence the coefficients of the fundamental quadratic form** are:

$$g_{ii} = (1 - k_i x^3)^{(2)} (g_{ii})_0 \qquad (i = 1,\ 2); \tag{3.6a}$$

$$g_{33} = 1;\ g_{ik} = 0 \qquad (i \neq k). \tag{3.6b}$$

where the $(g_{ii})_0$ are the coefficients of the basic quadratic form for $x^3 = 0$, i.e., on the basic surface.

Equations (3.6a) and (3.6b) allow us, as we shall show later, to find the metrics for a shell with an arbitrarily assigned coordinate net on its undeformed basic surface.

Section 4. Shells of Revolution. Special Cases of Shells of Revolution. Arbitrary Cylindrical Shells.

Consider a shell in which the basic surface is a surface of revolution. If the axis OZ of a rectangular Cartesian coordinate system is superposed on the axis of revolution of the basic surface, then the vector equation of the basic surface may be written in the following form:

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* See, for instance, W.Blaschke, Differential Geometry. ONTI, 1936

** Here and hereafter the exponents are written in parentheses. We will deviate from this rule in cases where it could not cause misunderstanding of the notation.

$$\vec{r}_0(x^1,\ x^2)=F(x^1)\,[\vec{i}\cos\varphi\,(x^2)+\vec{j}\sin\varphi\,(x^2)]+\vec{k}z(x^1). \tag{4.1}$$

where the equations

$$x = F(x^1),\ z = z(x^1) \tag{4.2}$$

determine the form of the meridional section of the shell, and the angle φ is a function of the second coordinate x^2 defining the position of a point on a circle of latitude. Thus,

$$\partial_1\vec{r}_0 = F'(x^1)\,[\vec{i}\cos\varphi\,(x^2) + \vec{j}\sin\varphi\,(x^2)] + \vec{k}z'(x^1), \tag{4.3a}$$

$$\partial_2\vec{r}_0 = F(x^1)\,[-\vec{i}\sin\varphi\,(x^2) + \vec{j}\cos\varphi\,(x^2)]\,\varphi'(x^2), \tag{4.3b}$$

where the prime denotes differentiation with respect to the corresponding ar- /22
gument.

Since meridional sections and circles of latitude are lines of curvature on a surface of revolution, let us make use of eqs.(3.6a). We obtain

$$g_{11} = (1 - k_1x^3)^{(2)}\,[F'^{(2)}(x^1) + z'^{(2)}(x^1)]; \tag{4.4a}$$

$$g_{22} = (1 - k_2x^3)^{(2)}F^{(2)}(x^1)\,\varphi'^{(2)}(x^2). \tag{4.4b}$$

The radii of curvature R_1 and R_2 are determined from eqs.(4.2). Let us consider certain special cases.

1. <u>The Circular Cylindrical Shell</u>

Equations (4.2) here take the form:

$$x = a = \text{const},\ z = z(x^1).$$

where a is the radius of the cylinder and x_1 is the coordinate defining the position of a point on the generatrix of the basic surface.

Consequently, $k_1 = 0$, $k_2 = \frac{1}{a}$.

It follows from eqs.(4.4a) and (4.4b) that

$$g_{11} = z'^{(2)}(x^1);\ g_{22} = (a - x^3)^{(2)}\varphi'^{(2)}(x^2). \tag{4.5}$$

If we put

$$z = x^1 \text{ and } \varphi = x^2,$$

then

$$g_{11} = 1;\ g_{22} = (a - x^3)^{(2)}. \qquad (4.6)$$

2. Circular Conical Shell

Let the basic surface of the shell be of the form of a circular truncated cone. Let x^1 be the distance of a point of the basic surface of the shell, measured along the generatrix, to the base of greater radius. Let r_1 and r_2 be the radii of the bases of the truncated cone. Assume that $r_1 > r_2$. Let H be the altitude of the cone, and γ the angle between the generatrix and the axis of revolution. Let the axes OX and OY lie in the plane of the base of radius r_1. Then eqs.(4.2) take the form:

$$x = F(x^1) = r_1 - x^1 \sin\gamma,\ z = x^1 \cos\gamma. \qquad (4.7)$$

From eqs.(4.4a) and (4.4b), remembering that $k_1 = 0$, we find

$$g_{11} = 1, \qquad (4.8a)$$

$$g_{22} = (1 - k_2 x^3)^{(2)} (r_1 - x^1 \sin\gamma)^{(2)} \varphi'^{(2)}(x^2). \qquad (4.8b)$$

where /23

$$k_2 = \frac{\cos\gamma}{r_1 - x^1 \sin\gamma}. \qquad (4.9)$$

3. The Shell with the Base Area in the Form of a Hyperboloid of Revolution

Consider a shell whose base area is the hyperboloid formed by the revolution of the hyperbola

$$\frac{x^{(2)}}{a^{(2)}} - \frac{z^{(2)}}{b^{(2)}} = 1 \qquad (4.10)$$

about the axis OZ. Equations (4.2) take the form of

$$x = F(x^1) = \frac{a}{b}\sqrt{b^{(2)} + (x^1)^{(2)}}\,;\ z = x^1. \tag{4.11}$$

Then,

$$k_1 = \frac{F''(x^1)}{[1 + F'^{(2)}(x^1)]^{3/2}}\,; \tag{4.12}$$

$$k_2 = \frac{1}{F(x^1)[1 + F'^{(2)}(x^1)]}. \tag{4.13}$$

The quantities g_{11}, g_{22} are defined by eqs.(4.4a) and (4.4b).

Consider, finally a shell with a cylindrical base area and an arbitrary directrix.

Let the coordinate x^1 define the distance of a point of the basic surface, measured along the generatrix, from one of the face sections, and the coordinate x^2 be equal to the length of the arc of a section of the basic surface by a plane normal to the generatrix, measured from one of the generatrices. Then the element of the arc of the basic surface will be

$$ds_0^2 = (dx^1)^{(2)} + (dx^2)^{(2)}. \tag{4.14}$$

Consequently,

$$(g_{11})_0 = (g_{22})_0 = 1,\ k_1 = 0; \tag{4.15}$$

and

$$g_{11} = 1;\ g_{22} = (1 - k_2 x^3)^{(2)}. \tag{4.16}$$

Section 5. <u>Scalars. Vectors and Their Contravariant and Covariant Components. The Reciprocal Coordinate Base</u>

Without dwelling on the properties and examples of scalar and vector quantities, familiar from physics and geometry, we will proceed to their analytical characterization.

We will apply the term absolute scalar to a quantity, determined by a function of the coordinates of the points in space, whose value at a fixed point in space does not depend on the choice of the coordinate system. In what follows we will also term such a quantity an invariant of coordinate transformations. The space in which a function defining an absolute scalar is assigned /24

is called a scalar field. In addition to absolute scalars there are also scalar quantities that do depend on the choice of the coordinate system. The projections of directed segments on the coordinate axes are examples of such quantities. When the term "scalar" is hereafter used, it will refer only to absolute scalars.

Consider now a certain vector $\vec{a}$, referred to the local coordinate base $\vec{e}_i$ of a curvilinear coordinate system. As is generally known, the vector $\vec{a}$ may be represented by the expansion (Bibl.7)

$$\vec{a} = \vec{e}_i a^i. \tag{5.1}$$

The quantities a^i are called the contravariant components of the vector $\vec{a}$. The meaning of this term will be explained below. In the general case the quantities a^i are functions of the coordinates of the points in space. The space in which the functions a^i are assigned is called the field of the vector $\vec{a}$.

To establish the analytic definition of the vector $\vec{a}$, consider the point transformation of the coordinates x^i and the change in the quantities a^i associated with this transformation. Let the formulas of transition from the coordinates x^i to the new coordinates y^i and from the new coordinates to the old be of the following form:

$$y^j = y^j(x^i), \tag{5.2a}$$

$$x^i = x^i(y^j) \tag{5.2b}$$

$$(i,\ j = 1,\ 2,\ 3).$$

Then the radius vector of an arbitrary point $M(y^i)$ may be regarded as a complex function of the x^i. By virtue of eq.(2.3) we obtain

$$\vec{e}_i = \frac{\partial \vec{r}}{\partial x^i} = \frac{\partial \vec{r}}{\partial y^j}\frac{\partial y^j}{\partial x^i} = \vec{e}\,'_j \frac{\partial y^j}{\partial x^i}. \tag{5.3a}$$

where the $\vec{e}\,'_j$ are the vectors of the new coordinate base. Equations (5.3a) are the formulas of transformation of the coordinate base. The formulas for the inverse transformation can be similarly found: /25

$$\vec{e}\,'_j = \vec{e}_i \frac{\partial x^i}{\partial y^j}. \tag{5.3b}$$

Equation (5.1) takes the form

$$\vec{a} = \vec{e}'_j \frac{\partial y^j}{\partial x^i} a^i = \vec{e}'_j a'^j; \tag{5.4}$$

where

$$a'^j = a^i \frac{\partial y^j}{\partial x^i}. \tag{5.5}$$

Equations (5.5) are the formulas of transformation of the quantities a^i. In exactly the same way we may find the formulas of the transformation inverse to eq.(5.5). Comparing the relations (5.3a) and (5.5), we conclude that the formulas of transformation of the quantities a^i are inverse in sense to the formulas for the transformation of the vectors of the coordinate base. Hence the term "contravariant".

Vectors can also be defined by a system of "generalized projections" onto the axes of the local coordinate base. Consider the three quantities

$$a_i = \vec{a} \cdot \vec{e}_i. \tag{5.6}$$

These quantities, likewise, analytically determine the vector $\vec{a}$. To convince ourselves that this is so, it is sufficient to express, in terms of the quantities a_i, the contravariant components of the vector $\vec{a}$.

By virtue of eqs.(5.1) and (2.5), we obtain*

$$a_i = \vec{e}_i \cdot \vec{e}_k a^k = g_{ik} a^k \qquad (i,\ k = 1,\ 2,\ 3). \tag{5.7}$$

Considering these equations as a system of linear algebraic equations in a^k, we find that

$$a^k = g^{ik} a_i \qquad (i,\ k = 1,\ 2,\ 3). \tag{5.8}$$

where

$$g^{ik} = \frac{1}{g} \frac{\partial g}{\partial g_{ik}}. \tag{5.9a}$$

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* Here and hereafter we make use of the right to denote the dummy indexes by any desired letter.

In orthogonal systems of coordinates:

$$g^{ii} = \frac{1}{g_{ii}}; \quad g^{ik} = 0 \qquad (i \neq k). \tag{5.9b}$$

After introduction of the quantities g^{ik}, eq.(5.1) assumes the form /26

$$\vec{a} = \vec{e}_k g^{ik} a_i. \tag{a}$$

We now introduce the notation

$$\vec{e}^i = g^{ik} \vec{e}_k. \tag{5.10}$$

Then eq.(a) takes the form

$$\vec{a} = \vec{e}^i a_i. \tag{5.11}$$

The vectors $\vec{e}^i$ form a coordinate base reciprocal to the original basis.

Let us consider a few relations, necessary for the further discussion, between the quantities introduced here. Compare the systems of equations (5.7) and (5.8). We have

$$a^k = g^{ik} g_{ij} a^j. \tag{b}$$

Since eqs.(b) are valid at arbitrary values of the quantities a^k (k,j = 1,2,3), the following identities hold:

$$g^{ik} g_{ij} = \delta^k_j = \begin{cases} 1 & (k = j), \\ 0 & (k \neq j). \end{cases} \tag{5.12}$$

where δ^k is the Kronecker symbol. Further, from eqs.(5.10), (2.5), and (5.12), we find

$$\vec{e}^i \cdot \vec{e}^k = g^{ij} g^{kl} \vec{e}_j \cdot \vec{e}_l = g^{ik}. \tag{5.13}$$

Equation (5.13) permits us to call the quantities g^{ik} coefficients of the

fundamental quadratic form on a reciprocal coordinate base. In exactly the same way we obtain

$$\vec{e}^i \cdot \vec{e}_k = \delta^i_k = g^i_{\cdot k}. \tag{5.14}$$

Obviously,

$$g^i_{\cdot k} = g_k^{\cdot i}.$$

Equations(5.14) permit us to find simple expressions of the vectors of a reciprocal coordinate base. We obtain

$$\vec{e}^i = \frac{e_j \times e_k}{V}. \tag{5.15a}$$

where × is the sign of the vector product.

The indices i, j, k form a cyclic permutation of the numbers 1, 2, 3.

Similarly,

$$\vec{e}_i = V(\vec{e}^j \times \vec{e}^k). \tag{5.15b}$$

The derivation of eqs.(5.15a) and (5.15b) from eqs.(5.14) is left to the reader. /27

We return now to eq.(5.1). On scalar multiplication of this equation by $\vec{e}^j$ and bearing eq.(5.14) in mind, we obtain

$$a^j = \vec{a} \cdot \vec{e}^j. \tag{5.16}$$

Consider the formulas of transformation of the quantities a_i. On the basis of eqs.(5.3a) and (5.6) we obtain

$$a_i = \vec{a} \cdot \vec{e}'_j \frac{\partial y^j}{\partial x_i} = a'_j \frac{\partial y^j}{\partial x_i}. \tag{5.17}$$

The formulas of the transformation inverse to eqs.(5.17) may be similarly found. These formulas allow us to call the quantities a_i the covariant components of the vector $\vec{a}$, since they coincide with the formulas of transformation of the coordinate vectors.

Conclusions

A scalar is a geometrical or physical quantity that is determined by a single function of a point in space and does not change its value at a fixed point under transformation of coordinates.

A vector is a geometrical or physical quantity determined by a system of three functions according to eqs.(5.1) and (5.11). These functions obey the transformation formulas (5.5) and (5.17). A characteristic feature of these formulas is their linearity and homogeneity relative to the coefficients of transformation (eq.5.2a):

$$\alpha_i^j = \frac{\partial y^j}{\partial x^i}. \tag{5.18}$$

The transformation formulas corresponding to eqs.(5.2b), as is readily demonstrated, are linear and homogeneous relative to the coefficients of the transformation*

$$\beta_j^i = \frac{\partial x^i}{\partial y^j}. \tag{5.19}$$

The transformation formulas are also homogeneous with respect to the vector components. For this reason, a vector equal to zero in one system of coordinates is equal to zero in all coordinate systems.

In concluding this Section, let us consider the projections of a vector /28
onto the axes of the local coordinate base. These projections are sometimes called "physical components of a vector" (Bibl.8).

Denoting the modulus of the vector $\vec{e}_i$ by e_i, we have

$$a_{xi} = \vec{a} \cdot \frac{\vec{e}_i}{e_i} = \frac{a_i}{\sqrt{g_{ii}}} = \frac{g_{ik}a^k}{\sqrt{g_{ii}}} \tag{5.20}$$

In orthogonal coordinate systems:

$$a_{xi} = a^i \sqrt{g_{ii}} \tag{5.21}$$

(do not sum over i!)

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* The expressions for the transformation coefficients adopted here are inverse to those used by us in an earlier book (Bibl.7).

Section 6. Tensors of Various Rank and Structure. The Metric Tensor of the Shell

The concept of the tensor is a natural generalization of the concepts of scalar and vector discussed above. The basis of this generalization is given by the formulas for transformation of vector components, eqs.(5.5) and (5.17). To find the direction of the generalizations, let us consider the formulas of transformation of the quantities g_{ik}, g^{ik}, and $g^{i\cdot}_{\cdot k}$. On the basis of the definitions of these quantities and of the transformation formulas for the vectors of the principal and reciprocal coordinate bases, we have

$$g'_{ik}=\vec{e}'_i\cdot\vec{e}'_k=\vec{e}_j\cdot\vec{e}_l\beta^j_i\beta^l_k, \tag{a}$$

or, finally,

$$g'_{ik}=\beta^j_i\beta^l_k g_{jl}. \tag{6.1}$$

Similarly,

$$g'^{ik}=\alpha^i_j\alpha^k_l g^{jl};\quad g'^{i\cdot}_{\cdot k}=\alpha^i_j\beta^l_k g^{j\cdot}_{\cdot l}. \tag{6.2}$$

A comparison of eqs.(6.1) and (6.2) with eqs.(5.5) and (5.17) leads to the wanted generalization.

A comparison of eqs.(6.1) and (6.2) with the vector component transformation formulas (5.5) and (5.17) and with the properties of scalars permits the inclusion of scalars, vectors, and of the quantities g_{ik}, g^{ik}, $g^{i\cdot}_{\cdot k}$ among the tensor quantities (tensors)*.

The structure of the relations (6.1) and (6.2) shows that we must distinguish tensors with covariant, contravariant, and mixed components. The difference between these components is that the formulas of transition from the old covariant components to the new contain only the coefficients β^i_j, the formulas of transition from the old contravariant components to the new contain only the coefficients α^j_i, while the formulas of transition from the old mixed components to the new contain the transformation coefficients α^j_i and β^i_j. /29

Further comparison of eqs.(6.1) and (6.2) with the relations (5.5) and (5.17) permits introduction of the concept of the rank of a tensor. The rank of a tensor is equal to the dimensionality of the right-hand sides of the transformational equations for its components relative to the coefficients of transformation.

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* The term "tensor" apparently originated in connection with the fact that the stresses in the neighborhood of a certain point of a continuous medium are components of the stress tensor. It is connected with the Latin word tendere, to pull, to stretch.

The number of components of a tensor depends on its rank. The number N of components of a tensor of rank n is expressed by the formula

$$N = 3^n. \qquad \text{(b)}$$

Thus scalars are tensors of zero rank, vectors are tensors of first rank, and the quantities g_{ik}, g^{ik}, $g^{i}_{\cdot k}$ are components of tensors of second rank. Tensors possessing the components g_{ik}, g^{ik} and $g^{i}_{\cdot k}$ are called metric, since they determine the measurement of distances between points of space and the measurement of angles between directed segments, i.e., the metric of space.

Generalizing eqs.(5.5), (5.17), (6.1) and (6.2), we set up a transformation formula for the components of a tensor of arbitrary structure and rank. These formulas are of the following form:

$$T'^{ik\cdots}_{\cdot\cdot j\cdots} = \alpha^i_p \alpha^k_q \beta^r_j \ldots T^{pq\cdots}_{\cdot\cdot r\cdots}. \qquad (6.3)$$

We advise the reader to set up the formula of the transformation inverse to eq.(6.3), as an exercise.

Equations (6.3) express the fundamental property of tensor components, the law of their transformation on passage from one coordinate system to another. This law is the same for all tensors, regardless of their geometrical meaning or physical nature. For this reason, to prove that any quantity has tensor properties, it is necessary and sufficient to prove that the transformation formulas (6.3) are satisfied. It follows from eqs.(6.3), in particular, that a tensor with components equal to zero in a certain system of coordinates will have components equal to zero in all systems of coordinates. In general, every tensor equation that is valid in one coordinate system will be satisfied in all other systems, i.e., such an equation will be invariant under transformation of coordinates.

If the components of a tensor in one system of coordinates are known, then eqs.(6.3) will permit us to find its components in any other system. In this case, the formulas of coordinate transformation (5.2a) and (5.2b) or the coefficients of transformation α^j_i and β^i_j must be assigned.

Thus, for example, we found the expressions (3.6a) and (3.6b) for the components of the metric tensor in the shell, under the assumption that the coordinate lines on the base surface coincide with its lines of curvature. These expressions permit a determination of the components of the metric tensor in /30 an arbitrary system of coordinates of the basic surface. Let the formulas of coordinate transformation be of the following form:

$$x^i = x^i(y^j) \quad (i,\ j = 1,\ 2), \qquad (6.4a)$$
$$x^3 = y^3. \qquad (6.4b)$$

Then, from eqs.(3.6a), (3.6b) and (6.1), we obtain

$$g'_{ij} = (1 - k_1 y^3)^{(2)} \frac{\partial x^1}{\partial y^i} \frac{\partial x^1}{\partial y^j} (g_{11})_0 +$$

$$+ (1 - k_2 y^3)^{(2)} \frac{\partial x^2}{\partial y^i} \frac{\partial x^2}{\partial y^j} (g_{22})_0; \quad (6.5a)$$

$$g'_{i3} = 0; \quad g'_{33} = 1 \qquad (i, j = 1, 2). \quad (6.5b)$$

In exactly the same way we could indicate the formulas of transformation of the components of the metric tensor, corresponding to an entirely arbitrary choice of the system of coordinates in the shell. However, we will not present them here.

Section 7. Operations of Tensor Algebra

Tensor algebra considers only those operations on tensors which result again in a tensor. It goes without saying that these operations do not include operations connected with differentiation or integration.

1. Addition

The operation of addition can be performed only on tensors of the same rank and structure.

The sum of tensors is the tensor determined by components equal to the sums of the components of the tensors being added:

$$T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} = A^{ik\cdot\cdot}_{\cdot\cdot j\cdot} + B^{ik\cdot\cdot}_{\cdot\cdot j\cdot} + C^{ik\cdot\cdot}_{\cdot\cdot j\cdot} + \dots \quad . \quad (7.1)$$

Indeed, if the quantities $A^{ik\cdot\cdot}_{\cdot\cdot j\cdot}$, $B^{ik\cdot\cdot}_{\cdot\cdot j\cdot}$, $C^{ik\cdot\cdot}_{\cdot\cdot j\cdot}$ are tensor components, i.e., if they obey the transformation formulas (6.3), then, obviously, the quantities $T^{ik\cdot\cdot}_{\cdot\cdot j\cdot}$ also obey the transformation formulas (6.3). This demonstrates that the operations defined by eq.(7.1) belong to tensor algebra.

2. Multiplication

The operation of multiplication may be applied to tensors of arbitrary rank and structure.

The product of tensors is the tensor with components equal to the products of the components of the tensors being multiplied. The rank of the product /31

equals the sum of the ranks of the factors. For example:

$$T^{ik\cdot\cdot}_{\cdot\cdot j\cdot\cdot} = A^{ik}B_{j\ldots} \tag{7.2}$$

If eqs.(6.3) are satisfied for the quantities A^{ik}, B_j,..., then it is obvious that they will also be satisfied for the quantities $T^{ik}_{\cdot\cdot j}{:}$. This demonstrates that the operations defined by the relation (7.2) belong to the operations of tensor algebra.

An example of the application of tensor multiplication is the construction of elementary, so-called multiplicative tensors. Assume, for instance, that we have assigned the vectors a^i, b^i, c_k*. The products of these quantities are the components of the mixed multiplicative tensor of third rank:

$$T^{ij\cdot}_{\cdot\cdot k} = a^i b^j c_k. \tag{7.3}$$

3. Contraction

The operation of contraction can be performed only on mixed tensors.

To perform this operation on the mixed tensor $T^{ik}_{\cdot\cdot j}{:}$ we set up the quantities

$$T^{i\cdot\cdot\cdot}_{\cdot\cdot\cdot} = T^{ij\cdot\cdot}_{\cdot\cdot j\cdot} \tag{7.4}$$

We shall prove that the quantities $T^{i}_{\cdot\cdot}{:}\cdot$ are components of a tensor having a rank two units lower than the rank of the original tensor, $T^{ik}_{\cdot\cdot j}{:}$.

Consider the transformation formula (6.3). Put in this formula k = j. In the right-hand side of the equation, the sum

$$\alpha^j_q \beta^r_j = \frac{\partial y^j}{\partial x^q}\frac{\partial x^r}{\partial y^j} = \delta^r_q = \begin{cases} 1 & (r = q), \\ 0 & (r \neq q). \end{cases} \tag{7.5}$$

will be eliminated. Thus, we obtain

$$T'^{ij\cdot\cdot}_{\cdot\cdot j\cdot\cdot} = \alpha^i_{p\ldots} T^{pq\cdots}_{\cdots q\cdots} \tag{a}$$

Consequently, the operation of contraction leads to a tensor of rank two units lower than the rank of the original tensor.

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* Here and hereafter the set of components of a tensor is, for brevity, itself called a tensor.

Example. Consider the mixed multiplicative tensor $a^i b_k$. By performing the operation of contraction on it, we obtain the scalar-scalar product of the vectors $\vec{a}$ and $\vec{b}$:

$$\vec{a}\cdot\vec{b}=a^i b_i=g^{ik}a_i b_k=g_{ik}a^i b^k. \tag{7.6}$$

4. "Raising" and "Lowering" of Indices

/32

We have already encountered special cases of this operation in discussing the relations (5.7) and (5.8). Let us extend it to tensors of any rank and structure.

We shall first show that an arbitrary tensor can be represented as the sum of multiplicative tensors. It is sufficient to demonstrate this in any specially selected coordinate system. Consider, to be definite, the third-rank tensor $T^{ij\cdot}_{\cdot\cdot k}$. Let us set up, corresponding to each component of this tensor, a system of three vectors $\overset{[ijk]}{a^p b^q c_r}$.

Let us select the vectors of this system, for instance, as follows:

$$a^p=\delta_i^p T^{ij\cdot}_{\cdot\cdot k},\quad b^q=\delta_j^q,\quad c_r=\delta_r^k.$$

Then the tensor $T^{ij\cdot}_{\cdot\cdot k}$ may be represented by the sum

$$T^{ij\cdot}_{\cdot\cdot k}=\sum_{p,q,r}\overset{[pqr]}{a^i b^j c_k}. \tag{b}$$

Bearing eq.(5.7) in mind, let us consider the equation

$$g_{il}T^{ij\cdot}_{\cdot\cdot k}=\sum_{p,q,r}g_{il}\overset{[pqr]}{a^i b^j c_k}=\sum_{p,q,r}\overset{[pqr]}{a_l b^j c_k}.$$

Thus,

$$g_{il}T^{ij\cdot}_{\cdot\cdot k}=T_{l\cdot k}^{\cdot j\cdot}. \tag{7.7}$$

We have "lowered" the first contravariant index, by converting it into a covariant index. A covariant index may similarly be "raised"

$$g^{kl}T^{ij\cdot}_{\cdot\cdot k}=T^{ijl}. \tag{7.8}$$

Consequently, any system of tensor components may be determined in a space with a given metric tensor, if a system of components of any structure is known.

We note in conclusion that the representation (b) of an arbitrary tensor by a sum of multiplicative tensors has a number of applications. For example, this representation permits us directly to find the "physical components" of an arbitrary tensor by making use of eqs.(5.20) and (5.21). For this it is sufficient to substitute for the components of the vectors $\overset{[p\,q\,r]}{a^i b^j c_k}$ their projections onto the axes of the local coordinate basis.

5. Permutation of Indices. Symmetrization and Alternation /33

The interchange of any pair of indices in the components of the tensor $T^{ik\cdot}_{\cdot\cdot j}$ transforms this tensor back into a tensor. If, on interchange of a pair of indices, the tensor remains unchanged, it is called symmetric with respect to this pair of indices. For example, on satisfying the condition

$$T^{ik\cdot}_{\cdot\cdot j} = T^{i\cdot k}_{\cdot j\cdot} \tag{7.9}$$

the tensor $T^{ik\cdot}_{\cdot\cdot j}$ is called symmetric with respect to the indices k and j. If, on interchange of a pair of indices, the components of the tensor change their signs, then the tensor is called antisymmetric with respect to this pair of indices. For example, on satisfying the condition

$$T^{ik\cdot}_{\cdot\cdot j} = -T^{i\cdot k}_{\cdot j\cdot} \tag{7.10}$$

the tensor $T^{ik\cdot}_{\cdot\cdot j}$ is antisymmetric with respect to the indices k and j.

Making use of the transformation formulas (6.3), it is easy to show that the properties of symmetry and antisymmetry are invariant under coordinate transformations (Bibl.7). The proof is left to the reader as an exercise.

A symmetric tensor of second rank has six substantially different components in three-dimensional space, while an antisymmetric tensor of second rank has only three. Indeed, we have, identically,

$$T^{ik\cdot}_{\cdot\cdot j} = \frac{1}{2}\left(T^{ik\cdot}_{\cdot\cdot j} + T^{i\cdot k}_{\cdot j\cdot}\right) + \frac{1}{2}\left(T^{ik\cdot}_{\cdot\cdot j} - T^{i\cdot k}_{\cdot j\cdot}\right). \tag{7.11}$$

The formation of a doubled symmetric part of a tensor is called symmetrization, and that of a doubled antisymmetric part is called alternation.

Section 8. Various Applications of Tensor Algebra

1. The Second Analytic Definition of the Tensor

We shall prove the following theorem: Given the system of quantities $T^{ik\cdot}_{\cdot\cdot j}$ and the arbitrary vectors a_i, b_k, c^j. If the sum $T^{ik\cdot}_{\cdot\cdot j} a_i b_k c^j$ is an invariant under transformation of coordinates (that is, an absolute scalar), then the quantities $T^{ik\cdot}_{\cdot\cdot j}$ are components of a mixed tensor of third rank.

Proof. By hypothesis, /34

$$T'^{pq\cdot}_{\cdot\cdot r}\, a'_p b'_q c'^r = T^{ik\cdot}_{\cdot\cdot j}\, a_i b_k c^j. \tag{a}$$

Let us make use of the vector component transformation formulas that result from eqs.(6.3). We have

$$a_i = \alpha_i^p a'_p; \quad b_k = \alpha_k^q b'_q; \quad c^j = \beta^j_r c'^r. \tag{b}$$

Substituting these relations into eq.(a) and transposing all terms to the left side of the equation, we obtain

$$(T'^{pq\cdot}_{\cdot\cdot r} - \alpha_i^p \alpha_k^q \beta_r^j T^{ik\cdot}_{\cdot\cdot j})\, a'_p b'_q c'^r = 0. \tag{c}$$

Equation (c) holds for arbitrary values of the quantities a'_p, b'_q and c'^r. This is possible only if all coefficients of the products $a'_p b'_q c'^r$ vanish. We then obtain

$$T'^{pq\cdot}_{\cdot\cdot r} = \alpha_i^p \alpha_k^q \beta_r^j T^{ik\cdot}_{\cdot\cdot j}. \tag{d}$$

We have again arrived at a relation of the form of eqs.(6.3). This proof may obviously be extended to a tensor of arbitrary rank and structure.

2. The Antisymmetric Tensor of Rank Two as a Vector in Three-Dimensional Space

We have already noted that an antisymmetric tensor of rank two in three-dimensional space has three substantially different components. We shall now show that there exists a vector equivalent to this tensor.

Let us first consider the transformation formulas for the vectors of the reciprocal coordinate base. From eqs.(5.10), (5.3b) and (6.2), we find

$$\vec{e}'^j = \alpha^j_q \vec{e}^q. \tag{8.1}$$

Turning now to eq.(5.15a), we have

$$\vec{e}'^{j} = \frac{\vec{e}'_i \times \vec{e}'_k}{V'}. \tag{e}$$

By using the transformation formulas (5.3b) we obtain

$$\vec{e}'^{j} = \frac{\beta_i^r \beta_k^s \vec{e}_r \times \vec{e}_s}{V'} = \frac{V}{V'} (\beta_i^r \beta_k^s - \beta_i^s \beta_k^r) \vec{e}^q. \tag{f}$$

Comparing the relations (8.1), (e) and (f), we obtain

$$\alpha_q^j = \frac{V}{V'} (\beta_i^r \beta_k^s - \beta_i^s \beta_k^r). \tag{8.2}$$

where the indices j, i, k and q, r, s take the values 1, 2, 3 in the order of /35
a positive cyclic permutation.

Consider the transformation formulas for the components of an antisymmetric covariant tensor of rank two. It follows from eqs.(6.3) and the antisymmetry of the tensor that

$$T'_{ik} = \beta_i^r \beta_k^s T_{rs} = \frac{1}{2} (\beta_i^r \beta_k^s - \beta_i^s \beta_k^r) T_{rs} \tag{g}$$

or, from eq.(8.2),

$$\frac{1}{V'} T'_{ik} = \frac{1}{2} \frac{1}{V} \sum_{r,s} \alpha_q^j T_{rs}. \tag{h}$$

The sign of summation in the right-hand side extends over all pairwise combinations of the three numbers 1, 2, 3, corresponding to the indices r and s.

We introduce the notation:

$$T^j = \frac{1}{\sqrt{g}} T_{ik} = -\frac{1}{\sqrt{g}} T_{ki}, \tag{8.3}$$

where the symbols j, i, k form the positive cyclic permutation of the numbers 1,

2, 3. Remembering eq.(2.6b), we obtain from (h)

$$T'^j = \alpha^j_q T^q. \tag{i}$$

This relation shows that eqs.(8.3) determine the components of the contravariant vector equivalent to the covariant antisymmetric tensor of rank two. It may be shown similarly that a contravariant antisymmetric tensor of rank two is equivalent to a covariant vector with the components

$$T_j = \sqrt{g}\, T^{ik} = -\sqrt{g}\, T^{ki}. \tag{8.4}$$

3. The Vector Product of Two Vectors in an Arbitrary Coordinate System

Consider the multiplicative tensor of rank two:

$$T_{ik} = a_i b_k. \tag{k}$$

Performing the operation of alternation, we obtain

$$R_{ik} = -R_{ki} = a_i b_k - a_k b_i. \tag{l}$$

The tensor R_{ik}, according to eqs.(8.3), is equivalent to the contravariant vector

$$c^j = \frac{1}{\sqrt{g}} (a_i b_k - a_k b_i). \tag{8.5}$$

The vector c^j determines the contravariant components of the vector product of the vectors $\vec{a}$ and $\vec{b}$ in an arbitrary coordinate system. Similarly, from eqs.(8.4), we obtain: /36

$$c_j = \sqrt{g}\, (a^i b^k - a^k b^i). \tag{8.6}$$

4. Pseudoscalars and Pseudovectors

Let us revert to eq.(2.6b). The volume V of the parallelepiped constructed on the vectors of the coordinate basis is a scalar. But it is impossible to find this scalar as an absolute. Under coordinate transformation, the volume V varies. In particular, on passage of the local coordinate basis from a right-hand system of coordinate vectors to a left-hand system, under preservation of the quantities of the vectors $\vec{e}_i$, the volume V changes sign. The volume V is therefore called a pseudoscalar. The quantities R_{ik}, defined by eqs.(l) have

similar properties (for details see Bibl.7).

The vector $\vec{c}$ with the components expressed by eqs.(8.5) and (8.6), has a dual meaning. If the sign $\sqrt{g}$ is fixed, then the components of the vector $\vec{c}$ will change signs on passage of the local coordinate system from a right-hand to a left-hand system. In this case the components of the vector $\vec{c}$ do not obey the transformation formulas (6.3) and the vector $\vec{c}$ is called a pseudovector.

If, however, $\sqrt{g}$ is regarded as a pseudoscalar, then eqs.(8.5) and (8.6) determine a polar vector, i.e. a vector that does obey the transformation law (6.3).

We note in conclusion that a vector product exists as a vector only in three-dimensional space. In multi-dimensional space it is considered an anti-symmetric tensor of rank two instead of a vector.

Section 9. The Absolute Differential of a Tensor. The Tensor Field and the Absolute Derivative

1. The Absolute Differential of a Tensor

Consider the variable vector $\vec{a}$ with contravariant components:

$$\vec{a} = \vec{e}_i a^i. \tag{a}$$

Assuming that the components of the vector and the points of its application /37 vary, we find the differential $d\vec{a}$:

$$d\vec{a} = \vec{e}_i da^i + a^i d\vec{e}_i. \tag{b}$$

Let us find the contravariant components of the differential $d\vec{a}$. From eqs.(5.16) we have

$$(d\vec{a})^j = da^j + a^i \vec{e}^j \cdot d\vec{e}_i. \tag{c}$$

Further,

$$d\vec{e}_i = \frac{\partial \vec{e}_i}{\partial x^k} dx^k. \tag{d}$$

Using eqs.(2.3) we now find

$$\frac{\partial \vec{e}_i}{\partial x^k} = \frac{\partial^2 \vec{r}}{\partial x^i \partial x^k} = \frac{\partial \vec{e}_k}{\partial x^i}. \tag{e}$$

We introduce the notation

$$\Gamma^j_{ik} = \vec{e}^j \cdot \frac{\partial \vec{e}_i}{\partial x^k} = \vec{e}^j \cdot \frac{\partial^2 \vec{r}}{\partial x^i \partial x^k}. \tag{9.1}$$

The quantities Γ^j_{ik} are called Christoffel symbols of the second kind. Christoffel symbols are symmetric with respect to the indices i, k:

$$\Gamma^j_{ik} = \Gamma^j_{ki}. \tag{9.2}$$

Equation (c) now takes the form:

$$(d\vec{a})^j = da^j + \Gamma^j_{ik} a^i dx^k. \tag{9.3}$$

Equation (9.3) determines the contravariant components of the absolute differential $d\vec{a}$ of the vector $\vec{a}$. The term "absolute differential" evidently arose in connection with the ideas of absolute motion, which are familiar from kinematics.

Consider the covariant components of the absolute differential. From eq.(5.11) we have

$$d\vec{a} = \vec{e}^i da_i + a_i d\vec{e}^i. \tag{f}$$

Let us find the covariant components of the vector $d\vec{a}$:

$$(d\vec{a})_j = da_j + a_i \vec{e}_j \cdot d\vec{e}^i. \tag{g}$$

It follows from eqs.(5.14) that:

$$\vec{e}_j \cdot d\vec{e}^i = -\vec{e}^i \cdot d\vec{e}_j. \tag{h}$$

Making use of the relations (d), (9.1) and (h), we obtain from eq.(g): /38

$$(d\vec{a})_j = da_j - \Gamma^i_{jk} a_i dx^k. \tag{9.4}$$

This relation determines the covariant components of the absolute differential of the vector $\vec{a}$.

We will now show that the Christoffel symbols are defined in terms of the components of the metric tensor, and indicate the formulas for their transformation. From eqs.(5.10) and (9.1) we have

$$\Gamma^j_{ik} = g^{js}\vec{e}_s \cdot \frac{\partial^2 \vec{r}}{\partial x^i \partial x^k} = g^{js}\Gamma_{s,ik}. \tag{9.5}$$

The quantities

$$\Gamma_{s,ik} = \vec{e}_s \cdot \frac{\partial^2 \vec{r}}{\partial x^i \partial x^k} = \vec{e}_s \cdot \frac{\partial e_i}{\partial x^k} \tag{i}$$

are called Christoffel symbols of the first kind. It follows from eqs.(9.5) and (5.12) that

$$\Gamma_{s,ik} = g_{rs}\Gamma^r_{ik}. \tag{9.6}$$

The following formula of transformation of the Christoffel symbols of the first kind results from eq.(i):

$$\Gamma'_{s,ik} = \Gamma_{p,qr} \frac{\partial x^p}{\partial y^s} \frac{\partial x^q}{\partial y^i} \frac{\partial x^r}{\partial y^k} + g_{qr} \frac{\partial x^r}{\partial y^s} \frac{\partial^2 x^q}{\partial y^i \partial y^k}. \tag{9.7a}$$

where the y^i are the coordinates of the new system.

From the transformation formula (9.7a) and eq.(9.6) it is easy to derive the transformation formula for Christoffel symbols of the second kind:

$$\Gamma'^j_{ik} = \Gamma^p_{rs} \frac{\partial y^j}{\partial x^p} \frac{\partial x^s}{\partial y^i} \frac{\partial x^r}{\partial y^k} + \frac{\partial y^j}{\partial x^p} \frac{\partial^2 x^p}{\partial y^i \partial y^k}. \tag{9.7b}$$

Thus, the Christoffel symbols are not components of a tensor, since they do not obey the transformation formulas (6.3).

Making use of the relation (e), we find

$$\Gamma_{s,ik} = \frac{1}{2}\left(\vec{e}_s \cdot \frac{\partial \vec{e}_i}{\partial x^k} + \vec{e}_s \cdot \frac{\partial \vec{e}_k}{\partial x^i}\right)$$

or

$$\Gamma_{s,ik} = \frac{1}{2}\left[\frac{\partial}{\partial x^k}(\vec{e}_i \cdot \vec{e}_s) + \frac{\partial}{\partial x^i}(\vec{e}_s \cdot \vec{e}_k) - \vec{e}_i \cdot \frac{\partial \vec{e}_s}{\partial x^k} - \vec{e}_k \cdot \frac{\partial \vec{e}_s}{\partial x^i}\right].$$

Again making use of the relation (e) and eq.(2.5), we obtain /39

$$\Gamma_{s,ik} = \frac{1}{2}\left(\frac{\partial g_{is}}{\partial x^k} + \frac{\partial g_{ks}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^s}\right). \qquad (9.8)$$

It follows from eq.(9.8) that

$$\frac{\partial g_{ir}}{\partial x^k} = \Gamma_{r,ik} + \Gamma_{i,rk}. \qquad (9.9a)$$

The absolute differential of a vector is thus completely determined if the metric of the space is known.

In conclusion we note the existence of a direct relation between the absolute differential of a vector and the absolute derivative of the vector function $\vec{a}(t)$, which we know from the principles of the kinematics of a rigid body*:

$$\frac{d\vec{a}}{dt} = \frac{d'\vec{a}}{dt} + \vec{\omega} \times \vec{a}. \qquad (9.9b)$$

where $\frac{d'\vec{a}}{dt}$ is the relative (local) derivative of the vector $\vec{a}$.

This interrelation results from eqs.(9.1) and from the definition of the instantaneous angular velocity $\vec{\omega}$ of the body**.

In this special case we find that between the Christoffel symbols and the instantaneous angular velocity of an absolutely rigid body there exists the relation:

$$\Gamma^j_{ik}\frac{dx^k}{dt} = \omega_i^{\cdot j} = -\omega^j_{\cdot i}, \qquad (9.9c)$$

where the $\omega_i^{\cdot j}$ are the components of the antisymmetric tensor of instantaneous angular velocity of the body, equivalent to the vector $\vec{\omega}$ (8.2)

A different interpretation of the meaning of the Christoffel symbols is also possible. It follows from relation (9.9c) that the product

$$(d\vec{\varphi})_i^{\cdot j} = \Gamma^j_{ik}\,dx^k \qquad (9.9d)$$

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* See (Bibl.7 p.130).

** See (Bibl.7 Sect.21)

determines the generalized relative angle of rotation of the coordinate basis under displacement from the point $M(x^i)$ to the neighboring point $M'(x^i + dx^i)$.

2. Absolute Differential of a Tensor of Arbitrary Rank and Structure

Consider the invariant

$$\varphi = T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} a_i b_k c^j \ldots \quad . \tag{k}$$

Differentiating the invariant φ, we obtain /40

$$d\varphi = dT^{ik\cdot\cdot}_{\cdot\cdot j\cdot} a_i b_k c^j + T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} b_k c^j da_i + \\ + T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} a_i c^j db_k + T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} a_i b_k dc^j + \ldots \quad . \tag{ℓ}$$

On the basis of eqs.(9.3) - (9.4) we represent eq.(ℓ) in the following form:

$$d\varphi = T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} b_k c^j (d\vec{a})_i \ldots + T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} a_i c^j (d\vec{b})_k \ldots + T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} a_i b_k (d\vec{c})^j \ldots + \\ + (dT^{ik\cdot\cdot}_{\cdot\cdot j\cdot} + T^{rk\cdot\cdot}_{\cdot\cdot j\cdot} \Gamma^i_{rs} dx^s + T^{ir\cdot\cdot}_{\cdot\cdot j\cdot} \Gamma^k_{rs} dx^s - \\ - T^{ik\cdot\cdot}_{\cdot\cdot r\cdot} \Gamma^r_{js} dx^s + \ldots) a_i b_k c^j + \ldots \quad . \tag{m}$$

Here we have changed the dummy indices necessary for the transformation of eq. (ℓ).

Considering eq.(m), we note that its left side and the first summands in its right side are scalars. Consequently, the last term in its right side is also a scalar. But the quantities a_i, b_k, c^j are components of arbitrary vectors. Consequently, according to the second analytic definition of a tensor (Sect.8), the expressions in parentheses are components of a mixed tensor:

$$DT^{ik\cdot\cdot}_{\cdot\cdot j\cdot} = dT^{ik\cdot\cdot}_{\cdot\cdot j\cdot} + \Gamma^i_{rs} T^{rk\cdot\cdot}_{\cdot\cdot j\cdot} dx^s + \Gamma^k_{rs} T^{ir\cdot\cdot}_{\cdot\cdot j\cdot} dx^s - \\ - \Gamma^r_{js} T^{ik\cdot\cdot}_{\cdot\cdot r\cdot} dx^s + \ldots \quad . \tag{9.10}$$

The tensor $DT^{ik\cdot\cdot}_{\cdot\cdot j\cdot}$, determined by eqs.(9.10) is called the absolute differential of the tensor $T^{ik\cdot\cdot}_{\cdot\cdot j\cdot}$.

3. Tensor Field. The Absolute (Covariant) Derivative of a Tensor of Arbitrary Rank and Structure

A tensor field is a region of variation of the coordinates x^i, such that to each point of the region there correspond values of the components of some tensor. We shall assume, with infrequent exceptions, that the components of the tensor are single-valued functions of the coordinates of the points of the field. We shall also assume that these functions have analytic singularities at isolated points of the field. At all other points of the field the tensor components are continuous and differentiable functions of the coordinates x^i. Then,

$$dT^{ik\cdot\cdot}_{\cdot\cdot j\cdot} = \partial_s T^{ik\cdot\cdot}_{\cdot\cdot j\cdot}\, dx^s. \tag{n}$$

Equation (9.10) now takes the form:

$$\begin{aligned} DT^{ik\cdot\cdot}_{\cdot\cdot j\cdot} &= (\partial_s T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} + \Gamma^i_{rs} T^{rk\cdot\cdot}_{\cdot\cdot j\cdot} + \Gamma^k_{rs} T^{ir\cdot\cdot}_{\cdot\cdot j\cdot} - \Gamma^r_{js} T^{ik\cdot\cdot}_{\cdot\cdot r\cdot} + \ldots)\, dx^s = \\ &= \nabla_s T^{ik\cdot\cdot}_{\cdot\cdot j\cdot}\, dx^s. \end{aligned} \tag{9.11}$$

The operator /41

$$\nabla_s T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} = \partial_s T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} + \Gamma^i_{rs} T^{rk\cdot\cdot}_{\cdot\cdot j\cdot} + \Gamma^k_{rs} T^{ir\cdot\cdot}_{\cdot\cdot j\cdot} - \Gamma^r_{js} T^{ik\cdot\cdot}_{\cdot\cdot r\cdot} + \tag{9.12}$$

is called the absolute or covariant derivative of the tensor $T^{ik\cdot\cdot}_{\cdot\cdot j\cdot}$. The geometrical meaning of absolute differentiation will be discussed in the following Section.

Let us return to eq.(9.9a). This relation may be represented in the following form:

$$\partial_k g_{ir} - \Gamma^s_{kr} g_{si} - \Gamma^s_{ki} g_{sr} = \nabla_k g_{ir} = 0. \tag{9.13}$$

Equation (9.13) expresses the theorem of Ricci: The absolute derivative of the metric tensor vanishes.

This assertion also applies to the contravariant and mixed components of the metric tensor (Bibl.7).

Consequently, in covariant differentiation the components of the metric tensor must be regarded as constant quantities. We suggest that the reader convince himself that the well-known rules for differentiation of the sum or product of scalar functions apply to the absolute differentiation of tensor functions.

Section 10. Parallel Displacement of Tensors in the Sense of Levi-Civita. The Tensor of Curvature.

1. Parallel Displacement

In the investigation of various vectors of geometry and mechanics, it is necessary to compare tensor quantities analytically assigned at different points of space.

This comparison can be accomplished after reduction of the quantities to be compared to a single point. We meet such reductions, in particular, in the kinematics and statics of an absolutely rigid body, where the system of sliding vectors is displaced parallel to their initial rectilinear bases to the center of reduction. In parallel displacement of a vector, neither its magnitude(modulus) nor its direction are changed. Consequently, in parallel displacement of a vector from the point $M(x^i)$ to the neighboring point $M'(x^i + dx^i)$, the absolute differential of the vector must vanish. Let us adopt the above statement as a general definition of parallel displacement of tensor quantities. This definition of parallel displacement coincides in essence with the definition given by Levi-Civita (Bibl.8).

Let $\delta T^{ik}_{..j..}$ be the change in the components of a tensor under parallel displacement from the point $M(x^i)$ to the neighboring point $M'(x^i + dx^i)$. Then, on the basis of eqs.(9.10), we can set up the system of differential equations of parallel displacement. This system has the following form: /42

$$\delta T^{ik..}_{..j.} = (-\Gamma^i_{rs} T^{rk..}_{..j.} - \Gamma^k_{rs} T^{ir..}_{..j.} + \Gamma^r_{js} T^{ik..}_{..r.} + \ldots)\, dx^s. \tag{10.1}$$

In particular, for a contravariant vector we find

$$\delta a^i = -\Gamma^i_{rs} a^r dx^s; \tag{10.2}$$

and for a covariant vector

$$\delta a_i = \Gamma^r_{is} a_r dx^s. \tag{10.3}$$

We note in conclusion that, in a vector displacement that is parallel in the Levi-Civita sense, the scalar product of the vectors remains unchanged and each of the vectors entering into the product may be independently displaced, and then the scalar product of the vectors so displaced can be constructed. The proof is left to the reader.

2. Tensor of Curvature (Riemann-Christoffel Tensor)

Equations (10.2) - (10.3) are not in general totally differential equations. The result of a parallel displacement of vectors in the Levi-Civita sense, therefore, depends on the shape and position of the curve along which

this displacement is accomplished. Parallel displacement in Euclidean space is here an exception. In this case it is always possible to choose a coordinate system such that the components of the metric tensor are constant and, consequently, the Christoffel symbols vanish.

Consider the result of a displacement of the contravariant vector a^i from the point $M(x^i)$ to the point $M''(x^i + dx^i + \delta x^i)$ on noncoinciding curves passing through the point $M'(x^i + dx^i)$ and $M_1(x^i + \delta x^i)$. Let us calculate the components of the displaced vector, using eq.(10.2). Let the components of the displaced vector at point M be a^i. Then, the components of the parallel-displaced vector at point M' will be

$$(a^i)_{M'} = (a^i)_M - (\Gamma^i_{jk})_M (a^j)_M \, dx^k. \tag{a}$$

The symbol M, here and hereafter, denotes the values of the functions at point M.

On further motion to point M'' we must bear in mind the change in the Christoffel symbols, which are functions of the coordinates of the points in space. We have, at point M''

$$(a^i)_{M''} = (a^i)_M + \Delta_1 a^i = (a^i)_M - (\Gamma^i_{jk})_M (a^j)_M \, dx^k -$$

$$- [(\Gamma^i_{pq})_M + (\partial_s \Gamma^i_{pq})_M \, dx^s] [(a^p)_M - (\Gamma^p_{jk})_M (a^j)_M \, dx^k] \, \delta x^q + \dots \, . \tag{b}$$

where the $\Delta_1 a^i$ are the changes in the components of the vector $\vec{a}$, on passage to the point M'' along the curve MM'M''. We shall neglect third-order infinitesimals. /43

Consider now the result of parallel displacement of the vector $\vec{a}$ to the point M'' along the curve MM_1M''. The components of the parallel-displaced vector at point M_1 will be expressed as follows:

$$(a^i)_{M_1} = (a^i)_M - (\Gamma^i_{jk})_M (a^j)_M \, \delta x^k. \tag{c}$$

At point M'',

$$(a^i)_{M''} = (a^i)_M + \Delta_2 a^i = (a^i)_M - (\Gamma^i_{jk})_M (a^j)_M \, \delta x^k -$$

$$- [(\Gamma^i_{pq})_M + (\partial_s \Gamma^i_{pq})_M \, \delta x^s] [(a^p)_M - (\Gamma^p_{jk})_M (a^j)_M \, \delta x^k] \, dx^q + \dots \, . \tag{d}$$

where the $\Delta_2 a^i$ are the changes in the vector components a^i on passage to the point M'' along the curve MM_1M''.

If the parallel displacement of the vector is accomplished in a space without internal curvature (in Euclidean space), then (b) and (d) are identically equal. Under parallel displacement of the vector in a non-Euclidean space, for instance on a nonplaner surface, (b) and (d) will not coincide. Consider the vector

$$\Delta a^i = \Delta_1 a^i - \Delta_2 a^i. \tag{e}$$

Subtracting (d) from (e), and making the necessary changes in the dummy indices, we obtain:

$$\Delta a^i = (\partial_r \Gamma^i_{jk} - \partial_k \Gamma^i_{jr} + \Gamma^i_{pr}\Gamma^p_{jk} - \Gamma^i_{pk}\Gamma^p_{jr})_M (a^j)_M \, dx^k \, \delta x^r. \tag{10.4}$$

Let us consider this equation. Noting that its left side contains contravariant components of the vector, we conclude, based on the second analytic definition of a tensor (Sect.8), that the expressions

$$R_{rk.j.}^{\cdot\cdot\cdot i} = \partial_r \Gamma_{jk} - \partial_k \Gamma^i_{jr} + \Gamma^i_{pr}\Gamma^p_{jk} - \Gamma^i_{pk}\Gamma^p_{jr} \tag{10.5}$$

are mixed components of a tensor of rank four. This tensor is called the curvature tensor, or the Riemann-Christoffel tensor.

In Euclidean space, the curvature tensor identically vanishes. In fact, in Euclidean space we can introduce a Cartesian coordinate system, in which all the Christoffel symbols vanish, as the components of the curvature tensor will then also vanish. However, a tensor that vanishes in one coordinate system will also vanish in all the others (Sect.6).

Consequently, in Euclidean space, the result of the parallel displacement of a vector, analytically determined in a curvilinear coordinate system, is independent of the choice of the curve along which the point of application of the vector is displaced. The vanishing of the tensor $R_{rk.j.}^{\cdot\cdot\cdot i}$ is a condition of integrability of the equations of parallel displacement.

The above discussion applies to the parallel displacement of tensors of arbitrary rank and structure. /44

Consider the elementary properties of the curvature tensor. It will be clear from eq.(10.5) that it is antisymmetric in the indices k and r. Let us find its covariant components. We have

$$R_{rj,ik} = g_{ks} R_{rj,i.}^{\cdot\cdot\cdot s} = g_{ks}(\partial_r \Gamma^s_{ji} - \partial_j \Gamma^s_{ir} + \Gamma^s_{pr}\Gamma^p_{ij} - \Gamma^s_{pj}\Gamma^p_{ir}). \tag{f}$$

Let us transform (f). From eq.(9.9a), we obtain

$$g_{ks}(\partial_r \Gamma^s_{ji} + \Gamma^s_{pr}\Gamma^p_{ij}) = \partial_r \Gamma_{k,ji} - \Gamma^s_{ji}(\Gamma_{k,sr} + \Gamma_{s,kr}) +$$

$$+ \Gamma_{k,pr}\Gamma^p_{ij} = \partial_r \Gamma_{k,ji} - \Gamma^s_{ij}\Gamma_{s,kr}.$$

Performing the operation of alternation with respect to the indices r and j, we obtain

$$R_{rj,ik} = \partial_r \Gamma_{k,ji} - \partial_j \Gamma_{k,ir} + \Gamma^s_{ir}\Gamma_{s,jk} - \Gamma^s_{ij}\Gamma_{s,kr}, \tag{10.6}$$

and, making use of the expressions (9.8) for the Christoffel symbols, we get

$$R_{rj,ik} = \frac{1}{2}\left(\frac{\partial^2 g_{kj}}{\partial x^i \partial x^r} + \frac{\partial^2 g_{ir}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ji}}{\partial x^k \partial x^r} - \frac{\partial^2 g_{kr}}{\partial x^j \partial x^i}\right) +$$

$$+ g^{qs}\Gamma_{q,ir}\Gamma_{s,jk} - g^{qs}\Gamma_{q,ij}\Gamma_{s,kr}. \tag{10.7}$$

This formula gives us the fundamental properties of the curvature tensor, its antisymmetry in the indices r, j and i,k and its symmetry in the index-pairs rj and ik. Hence follows, more specifically, that in a three-dimensional space the curvature tensor has only six substantially different components, and in two-dimensional space (on a nonplanar surface), one.

In Euclidean space, as already noted, the curvature tensor vanishes. Its vanishing is a necessary and sufficient condition for the possibility of introducing into a space a system of coordinates with the Euclidean metric, in which the components of the metric tensor are expressed by the equations:

$$g_{ik} = \delta^i_k. \tag{10.8}$$

We shall not dwell here on the proof of this assertion, nor on the study of the various properties of the curvature tensor, and refer the reader to the specialized manuals*.

3. Change of the Sequence of Operations in Successive Absolute Differentiation /45

It can be shown that a change in the sequence of operations of covariant differentiation substantially changes the result in cases where the curvature tensor does not vanish.

The following equality can be proved by direct calculation:

* Cf., for instance, Rashevskiy, P.K., Riemannian Geometry and Tensor Analysis. Gostekhizdat, 1953.

$$\nabla_i \nabla_k u^j - \nabla_k \nabla_i u^j = R_{ik,\ r}^{\cdot\cdot\ j} u^r. \tag{10.9}$$

Consequently, repeated covariant differentiation is commutative in Euclidean space.

4. Geometric Construction of a Covariant Derivative

We can now convince ourselves that the absolute derivative determines the major part of the increment of a tensor function, like the derivative that determines the major part of the increment of a scalar function. Consider, for instance, the components of the increment of a contravariant vector corresponding to the difference between the coordinates of point $M(x^i)$ and $N(x^i + dx^i)$. To construct the vector increment, having vector properties, at point M or point N, we must use the operation of parallel displacement. We have

$$(\Delta \vec{a})^i_N \approx a^i_M + (\partial_s a^i)_M dx^s - [a^i_M - (\Gamma^i_{rs})_M a^r_M dx^s] =$$

$$= (\partial_s a^i)_M + (\Gamma^i_{rs})_M a^r_M dx^s = (\nabla_s a^i)_M dx^s.$$

This equality compels attention to the duality in the meaning of the result: the constructed quantities $(\Delta\vec{a})^i_N$ do have the properties of a vector at point N, but are expressed in terms of tensor quantities determined at point N.

Section 11. Operator of Parallel Displacement of Tensor Quantities on the Base Area of a Shell

Most studies on shell theory are based on the reduction of the three-dimensional problems of the theory of elasticity and plasticity to two-dimensional problems, by means of the analytic determination of the quantities sought in the coordinates and metric of the base surface.

We shall therefore now discuss the problem of the parallel displacement of tensor quantities from an arbitrary point on a given shell to the base surface. This displacement may be accomplished by integrating eqs.(10.1). /46

We shall here consider the integration of the simpler formulas [eqs.(10.2) - (10.3)], which permit the parallel displacement of both contravariant and covariant vectors. We shall use the method of successive approximation employed by us elsewhere (Bibl.23b) for this purpose.

Let us consider again eqs.(10.2) for the parallel displacement of a contravariant vector, and represent them in the following form:

$$da^i = -\Gamma^i_{rs} a^r dx^s. \tag{a}$$

Given the components of the vector a^i at some point $M(x^i_0)$ of the shell. Let us denote these components by a^i_0. Required, to find the components a^i after

the parallel displacement of the point of application of the vector along an arbitrary curve to the base surface.

Assume that the equations of the displacement curve are of the form

$$x^i = x^i(u), \tag{11.1}$$

and that

$$x_0^i = x^i(u_0); \quad a_0^i = a^i(x_0^i) = a^i(u_0). \tag{b}$$

Then, on the curve selected by us, the Christoffel symbols will be assigned functions of the parameter u. Then, eqs.(a) will take the form

$$da^i = M_r^i a^r du, \tag{11.2}$$

where

$$M_r^i(u) = -\Gamma_{rs}^i(u)\dot{x}^s(u). \tag{11.3}$$

where the dot indicates differentiation with respect to u. Following my earlier work (Bibl.23b), we replace the system of equations (11.2) by the system of equivalent integral equations:

$$a^i(u, u_0) = a_0^i + \int_{u_0}^{u} M_r^i(v)\, a^r(v)\, dv. \tag{11.4}$$

Substituting the initial values of the components of the vector a^i in the expression under the sign of integration, we obtain the first approximation:

$$a_{(1)}(u, u_0) = a_0^i + a_0^r \int_{u_0}^{u} M_r^i(v)\, dv. \tag{c}$$

Substituting, again, the first approximation (c) into the expression under the integral sign in the right side of eq.(11.4), we find

$$a_{(2)}^i(u, u_0) = a_0^i + a_0^r \int_{u_0}^{u} M_r^i(v)\, dv + a_0^s \int_{u_0}^{u}\int_{u_0}^{v} M_s^r(v_1)\, M_r^i(v)\, dv_1\, dv. \tag{d}$$

Continuing this process, we obtain, after several permutations of the indices, /47

$$a^i(u, u_0) = a_0^i + a_0^r \Phi_r^i(u, u_0). \tag{11.5}$$

where

$$\Phi_r^i(u, u_0) = \int_{u_0}^{u} M_r^i(v)\, dv + \int_{u_0}^{u}\int_{u_0}^{v} M_r^s(v_1)\, M_s^i(v)\, dv_1\, dv + \ldots =$$

$$= \int_{u_0}^{u} M_r^i(v)\, dv + \int_{u_0}^{u}\int_{v_1}^{u} M_r^s(v_1)\, M_s^i(v)\, dv\, dv_1 + \ldots \,. \tag{11.6}$$

Equation (11.6) defines the resolvent of the system of integral equations (11.4). The displacement of a covariant vector may be similarly considered. We have, from eqs.(10.3),

$$da_i = N_i^r a_r\, du, \tag{11.7}$$

where

$$N_i^r(u) = \Gamma_{is}^r(u)\, \dot{x}^s(u). \tag{11.8}$$

From eq.(11.7) we find

$$a_i(u, u_0) = a_{i0} + \int_{u_0}^{u} N_i^r(v)\, a_r(v)\, dv. \tag{11.9}$$

This system of integral equations, like the system (11.4), is solved by the method of successive approximation. Its solution is of the form:

$$a_i(u, u_0) = a_{i0} + a_{r0}\, \Psi_i^r(u, u_0). \tag{11.10}$$

The resolvent $\Psi_i^r(u, u_0)$ is expressed as follows:

$$\Psi_i^r(u, u_0) = \int_{u_0}^{u} N_i^r(v)\, dv + \int_{u_0}^{u}\int_{u_0}^{v} N_s^r(v_1)\, N_i^s(v)\, dv_1 dv + \ldots =$$

$$= \int_{u_0}^{u} N_i^r(v)\, dv + \int_{u_0}^{u}\int_{v_1}^{u} N_s^r(v_1)\, N_i^s(v)\, dv\, dv_1 + \ldots \,. \quad (11.11) \tag{11.11}$$

The proof for the convergence of these expansions is known from the theory of Volterra's integral equations of the second kind. With insubstantial re-

strictions I have also presented the proof of convergence in the work already cited (Bibl.23b).

The resolvents $\Phi_r^i(u, u_0)$ and $\Psi_i^r(u, u_0)$, like the Christoffel symbols, are not tensor quantities. These operators permit the displacement of tensor quantities over a finite distance. We shall call them operators of parallel displacement.

The formulas (11.5) and (11.10) can be put into a different form. By setting /48

$$A_r^i(u, u_0) = \delta_r^i + \Phi_r^i(u, u_0); \quad B_i^r(u, u_0) = \delta_i^r + \Psi_i^r(u, u_0), \tag{11.12}$$

we get

$$a^i(u, u_0) = A_r^i(u, u_0)\, a^r(u_0); \quad a_i(u, u_0) = B_i^r(u, u_0)\, a_r(u_0). \tag{11.13}$$

The relations (11.5) and (11.10) written in this form are analogous to the vector-component transformation formulas derived from eqs.(6.3). We therefore extend eqs.(11.13) to a tensor of arbitrary rank and structure. By analogy to eqs.(6.3) we obtain

$$T^{ik\cdot\cdot}_{\cdot\cdot j\cdot}(u, u_0) = A_p^i(u, u_0)\, A_q^k(u, u_0)\, B_j^r(u, u_0) \ldots T^{pq\cdot\cdot}_{\cdot\cdot r\cdot}(u_0). \tag{11.14}$$

Let us consider, as an example, the construction of the operators $\Phi_r^i(u, u_0)$ and $\Psi_i^r(u, u_0)$. Let the metric of the shell be expressed by eqs.(3.6a)-(3.6b). The metric defined by these equations is encountered in an undeformed shell, or in a deformed shell if - after its deformation - we choose a new coordinate system with the coordinate lines coinciding with the lines of curvature on the deformed base surface and with the normals to it.

Assume, for simplicity, that the displacement takes place along a normal to the base surface. Then, eqs.(11.1) can be put into the form

$$x^1 = x_0^1; \quad x^2 = x_0^2; \quad x^3 = u. \tag{11.15}$$

Let us also put:

$$u_0 = z. \tag{11.16}$$

In this case:

$$\dot{x}^1 = \dot{x}^2 = 0; \quad \dot{x}^3 = 1, \tag{e}$$

then

$$M_r^i(u) = -\Gamma_{r3}^i(u); \quad N_r^i(u) = \Gamma_{r3}^i(u). \tag{f}$$

Bearing formulas (5.9b), (9.5) and (9.8) in mind, and calculating the Christoffel symbols Γ^i_{r3}, we find that only the symbols Γ^i_{i3} do not vanish in the system of coordinates we have selected. We obtain

$$\Gamma^i_{i3} = \frac{1}{2g_{ii}}\partial_3 g_{ii} = \partial_3 \ln(1-k_i x^3) \qquad (i=1;\ 2), \tag{11.17}$$

or, in view of eqs.(11.15),

$$\Gamma^i_{i3}(u) = \frac{\partial}{\partial u}\ln(1-k_i u). \tag{g}$$

Making use of eqs.(f) and (g) and the expression for the resolvent, (11.6), we find /49

$$\Phi^i_i(u, u_0) = \ln\frac{1-k_i u_0}{1-k_i u} + \frac{1}{2!}\ln^2\frac{1-k_i u_0}{1-k_i u} + \ldots = e^{\ln\frac{1-k_i u_0}{1-k_i u}} - 1, \tag{h}$$

or, in the notation of eqs.(11.15) - (11.16),

$$\Phi^i_i(x^3, z) = \frac{k_i(x^3-z)}{1-k_i x^3} \qquad (i=1,\ 2). \tag{11.17}$$

The remaining operators Φ^i_r vanish. For $x^3 = 0$, eq.(11.17) yields the operator of parallel displacement of a contravariant vector to the base surface (Bibl.23b):

$$\Phi^i_i(0, z) = -k_i z. \tag{11.18}$$

Let us now determine the operator $\Psi^r_i(u, u_0)$. Using eqs.(f) and (g) and the expression for the resolvent, eq.(11.11), we find

$$\Psi^i_i(u, u_0) = -\ln\frac{1-k_i u_0}{1-k_i u} + \frac{1}{2!}\ln^2\frac{1-k_i u_0}{1-k_i u} - \ldots = e^{-\ln\frac{1-k_i u_0}{1-k_i u}} - 1, \tag{i}$$

or

$$\Psi^i_i(x^3, z) = \frac{k_i(z-x^3)}{1-k_i z} \qquad (i=1,\ 2). \tag{11.19}$$

The other operators Ψ^r_i vanish. For $x^3 = 0$, eq.(11.19) gives the operator of parallel displacement of a covariant vector to the base surface:

$$\Psi^i_i(0, z) = \frac{k_i z}{1 - k_i z}. \tag{11.20}$$

Equations (11.17) and (11.19) can be directly obtained from the system of equations (11.2) and (11.7), since when the relations (f) and (g) are satisfied, the system of equations of parallel displacement breaks down into individual equations.

Section 12. Expansion of Tensor Functions in Generalized Taylor Series

1. Analytical Definition of the Radius Vector of a Point of Space in Curvilinear Coordinates

In analytic geometry the term radius vector is customarily applied to a directed segment drawn from a fixed point (the origin of coordinates) to a point in space. In a Cartesian coordinate system, the contravariant components of /50 the radius vector of a point are equal to the components of its terminus or to differences between the coordinates of the terminus of the radius vector and those of its fixed origin. Thus, the radius vector is a geometrical object connected with two points in space, and therefore it is not a vector attachment, defined instead by the coordinates of its point of application. This causes the trouble in attempts at analytic definition of the radius vector in curvilinear systems of coordinates, since the transformation formula (6.3) relates to a fixed point in space.

To avoid misunderstandings, we shall introduce the radius-vector into systems of curvilinear coordinates by means of definition. We shall first define the radius vector in the Cartesian system of coordinates, as just indicated. We shall then define its contravariant components in an arbitrary curvilinear system of coordinates, applying the transformation formulas (6.3). We shall at the same time also define the transformation coefficients at the fixed origin of the radius vector. Obviously a radius vector can exist only in a space that permits introduction of the Cartesian coordinates. It does not exist in the internal geometry of nonplanar surfaces. Here we can introduce only small radius vectors with errors of the second order of smallness.

2. Expansion of Tensor Functions into Generalized Taylor Series

The three-dimensional problems of the theory of elasticity and plasticity are reduced to two-dimensional problems by various methods, among which we must mention the method given by Cauchy and Poisson in the theory of plates. This method, based on the expansion of the required quantities into Taylor series, will be discussed in Chapter III. Here we shall dwell only on the general prop-

erties of such expansions in the space within a shell, referred to curvilinear coordinates.

Let us first consider the tensor $\overset{n\to}{T}$ of rank n, referred to the Cartesian system of coordinates. Expanding the components of this tensor in Taylor series in the neighborhood of some fixed point M, in powers of the coordinate increments, and returning again to the non-coordinate representation of tensor quantities, we find

$$\overset{n\to}{T}_N = \overset{n\to}{T}_M + \Delta \overset{n\to}{T}_M + \frac{1}{2!}\Delta^2 \overset{n\to}{T}_M + \dots \quad . \tag{12.1}$$

where the letters M and N denote quantities determined at the fixed points M and N.

In the expanded form, in the Cartesian coordinate system, eq.(12.1) has the form: /51

$$\begin{aligned}(T^{ik\cdot\cdot}_{\cdot\cdot j\cdot})_N = (T^{ik\cdot\cdot}_{\cdot\cdot j\cdot})_M + (\partial_p T^{ik\cdot\cdot}_{\cdot\cdot j\cdot})_M (\Delta\vec{r})^p +\\ + \frac{1}{2!}(\partial_p \partial_q T^{ik\cdot\cdot}_{\cdot\cdot j\cdot})_M (\Delta\vec{r})^p (\Delta\vec{r})^q + \dots \quad .\end{aligned} \tag{12.2}$$

where $\Delta\vec{r}$ is the radius vector with its origin at the point M and its terminus at the point N.

On passage to curvilinear coordinates, the derivatives ∂_s in eq.(12.2)must be replaced by the absolute derivatives $\nabla_{(s)}$. We find

$$\begin{aligned}(T^{ik\cdot\cdot}_{\cdot\cdot j\cdot})_{N|M} = (T^{ik\cdot\cdot}_{\cdot\cdot j\cdot})_M + (\nabla_p T^{ik\cdot\cdot}_{\cdot\cdot j\cdot})_M (\Delta\vec{r})^p +\\ + \frac{1}{2!}(\nabla_p \nabla_q T^{ik\cdot\cdot}_{\cdot\cdot j\cdot})_M (\Delta\vec{r})^p (\Delta\vec{r})^q + \dots \quad .\end{aligned} \tag{12.3}$$

This equation defines the expansion of the tensor $\overset{n\to}{T}$ of parallel displacement from point N to point M. In other words, this expansion defines the components of the tensor $\overset{n\to}{T}$ at point N in terms of the values of these components and their derivatives at point M and in the metric of space at point M. The proof of eq.(12.3) follows from two propositions:

a) On passage to a Cartesian system of coordinates, eq.(12.3) is transformed into eq.(12.2), which results from the classical Taylor expansion.

b) A tensor equation valid in any system of coordinates is valid in all other systems.

CHAPTER II /52

PRINCIPAL RELATIONS OF THE NONLINEAR THEORY OF ELASTICITY IN THE INVARIANT FORM

Section 1. Euler and Lagrange Variables. Displacement Vector, Velocity Vector and Acceleration Vector of an Element of a Continuous Medium

An arbitrary system of curvilinear coordinates, determining the position of points of a continuous medium, but not connected with the medium, is called a system of Euler variables. The Euler variables of the points of a continuous medium vary on its motion.

A system of coordinates determining the position of points of a medium and materially connected with that medium is called a system of Lagrange variables. The Lagrange variables of the points of a medium do not vary on its motion.

Let us assume for simplicity that the Euler variables are the Cartesian coordinates ξ_i, while the Lagrange variables are the arbitrary curvilinear coordinates x^i. The quantities x^i likewise determine a certain Eulerian coordinate system. This will be discussed later in Sections 2 and 3.

Let us introduce in the space of Eulerian coordinates a radius vector determining the position of the points of the medium. When the position of the points of the medium varies, the radius vector of a certain point $M(x^i)$ will also vary. We have

$$\vec{r}(t,\ x^i) = \vec{r}(0,\ x^i) + \vec{u}(t,\ x^i) \qquad (i = 1,\ 2,\ 3). \tag{1.1}$$

The increment $\vec{u}(t,\ x^i)$ of the radius vector $\vec{r}(0,\ x^i)$, determining the initial positions of a point of the continuous medium, is called the displacement vector of the point $M(x^i)$. The vector $\vec{u}(t,\ x^i)$ is a function of the Lagrangian coordinates x^i and the time t.

Determining the components of the radius vector $\vec{r}(t,\ x^i)$ in Eulerian coordinates, we obtain

$$\xi_i(t,\ x^i) = \xi_i(0,\ x^i) + u_{\xi_i}(t,\ x^i). \tag{1.2}$$

where the u_{ξ_i} are the "physical components" of the vector $\vec{u}(t,\ x^i)$ in Eulerian coordinates. Equations (1.2) may be considered as formulas of transition with the parameter t, connecting the Lagrangian and Eulerian coordinates. /53

By differentiating eq.(1.1) with respect to t, we find the velocity vector and the acceleration vector of an element of the continuous medium:

$$\vec{v} = \frac{\partial \vec{r}}{\partial t} = \frac{\partial \vec{u}}{\partial t}; \quad \vec{w} = \frac{\partial^2 \vec{r}}{\partial t^2} = \frac{\partial^2 \vec{u}}{\partial t^2}. \tag{1.3}$$

Section 2. Tensor of Small Deformations and Tensor of Finite Deformations

1. Tensor of Small Deformations and Vector of Small Rotation of an Element of a Continuous Medium

Let us return to eq.(1.1). This equation permits the introduction of the fundamental quantities describing the variation of the internal geometrical properties of a space invariably bound to the deformable medium. Such quantities are the tensor of small deformations and the tensor of finite deformations. Let us consider first the tensor of small deformations.

Differentiating eq.(1.1) with respect to the coordinates x^i, we find

$$d\vec{r}(t, x^i) = d\vec{r}(0, x^i) + d\vec{u}(t, x^i). \tag{a}$$

Further,

$$d\vec{r}(0, x^i) = \partial_k \vec{r}(0, x^i)\, dx^k = \vec{e}_{k0}\, dx^k. \tag{b}$$

where $\vec{e}_{k0}$ are the vectors of the local coordinate basis in the undeformed medium.

Let us continue the transformation of eq.(a). Using eqs.(I, 9.3) and (I, 9.11), we obtain

$$d\vec{u}(t, x^i) = \vec{e}_{j0}\,(d\vec{u})^j = \vec{e}_{j0} \nabla_k u^j dx^k. \tag{c}$$

where the covariant derivative is determined in the metric of the undeformed medium. Consequently,

$$d\vec{r}(t, x^i) = (\vec{e}_{k0} + \vec{e}_{j0} \nabla_k u^j)\, dx^k. \tag{2.1}$$

Denoting the contravariant components of the vector $d\vec{r}$ by dx'^i, we get

$$dx'^i = (\delta^i_k + \delta^i_j \nabla_k u^j)\, dx^k. \tag{2.2}$$

Equations (2.2) show that the deformation of a continuous medium may be regarded to be a result of local transformations of coordinates in the neighborhoods of the points of the medium.

The transformation coefficients /54

$$a_k^i = \delta_k^i + \delta_j^i \nabla_k u^j \tag{2.3}$$

are components of a mixed tensor of rank two in the metric of the undeformed medium. Let us consider the tensor

$$\Phi_{k.}^{\cdot i} = \delta_j^i \nabla_k u^j = \nabla_k u^i. \tag{2.4}$$

The tensor $\overset{2}{\vec{\Phi}}$ is called the differential expansion of the vector $\vec{u}$(Bibl.7). We introduce the covariant components of the tensor $\overset{2}{\vec{\Phi}}$ and expand this tensor into its symmetric and antisymmetric parts (I, 7.5). We find

$$\Phi_{ki} = \nabla_k u_i = \frac{1}{2}(\nabla_k u_i + \nabla_i u_k) + \frac{1}{2}(\nabla_k u_i - \nabla_i u_k). \tag{2.5}$$

The symmetric tensor

$$\varepsilon_{ki} = \varepsilon_{ik} = \frac{1}{2}(\nabla_k u_i + \nabla_i u_k) = \frac{1}{2}(\partial_i u_k + \partial_k u_i - 2\Gamma_{ik}^j u_j) \tag{2.6}$$

is called the tensor of small deformations of an element of the continuous medium. The meaning of this term will be explained below.

The antisymmetric tensor

$$\Omega_{ki} = -\Omega_{ik} = \frac{1}{2}(\nabla_k u_i - \nabla_i u_k) \tag{2.7}$$

leads, on the basis of (I, 8.2) to the vector

$$\Omega^j = \frac{1}{2\sqrt{g}}(\nabla_k u_i - \nabla_i u_k) = \frac{1}{2\sqrt{g}}(\partial_k u_i - \partial_i u_k). \tag{2.8}$$

The indices j, k, i are a cyclic permutation of the numbers 1, 2, 3. The vector $\vec{\Omega}$ is called the curl of the vector $\vec{u}$:

$$\vec{\Omega} = \mathrm{curl}\, \vec{u}. \tag{2.9}$$

It is well known that the vector $\vec{\Omega}$ approximately determines the absolute rotary displacement of the particles of the medium (Bibl.7)*.

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* The identification of the vector $\vec{\Omega}$ with the mean angle of rotation is possible only in the linear theory (Bibl.11b).

On the basis of (I, 9.9d) we note that the generalized relative angle of rotation of adjacent elements of the deformed medium is expressed in terms of the Christoffel symbol in this medium. To obtain a complete idea of the kinematics of a medium after deformation, one must turn to the investigation of its metric.

2. Tensor of Finite Deformations /55

To find the kinematic quantity characterizing the change of the distance between two points of a continuous medium under deformation, and the change in the angle between the direction of two vectors $d\vec{r}_0$ and $\delta\vec{r}_0$ originating at the arbitrary point $M(x^i)$ of an undeformed medium under deformation, let us consider the change in the scalar product $d\vec{r}_0 \cdot \delta\vec{r}_0$ caused by deformation. We have, on the basis of eq.(b):

$$d\vec{r}_0 = \vec{e}_{i0} dx^i; \quad \delta\vec{r}_0 = \vec{e}_{k0}\delta x^k$$

and

$$d\vec{r}_0 \cdot \delta\vec{r}_0 = g_{ik} dx^i \delta x^k. \tag{d}$$

where the components of the metric tensor relate to the undeformed state of the medium. Further, by the aid of eq.(2.1), we obtain

$$d\vec{r} \cdot \delta\vec{r} = (\vec{e}_{i0} + \vec{e}_{j0} \nabla_i u^j)(\vec{e}_{k0} + \vec{e}_{r0}\nabla_k u^r)\, dx^i \delta x^k =$$

$$= (g_{ik} + g_{jk}\nabla_i u^j + g_{ir}\nabla_k u^r + g_{ir}\nabla_i u^j \nabla_k u^r)\, dx^i \delta x^k. \tag{2.10}$$

Finally, making use of the Ricci theorem (I, Sect.9.3), we find

$$d\vec{r} \cdot \delta\vec{r} - d\vec{r}_0 \cdot \delta\vec{r}_0 = (\nabla_i u_k + \nabla_k u_i + \nabla_i u^j \nabla_k u_j)\, dx^i \delta x^k. \tag{e}$$

The expressions in parentheses are the covariant components of the symmetric tensor of rank two:

$$2D_{ik} = \nabla_i u_k + \nabla_k u_i + \nabla_i u^j \nabla_k u_j. \tag{2.11}$$

Equations (2.11) determine the tensor of finite deformations of the continuous medium. From a comparison of eqs.(2.11) and (2.6) follows the following relation:

$$2D_{ik} = 2\varepsilon_{ik} + \Phi_{i\cdot}^{\cdot j} \Phi_{kj}. \tag{f}$$

For small values of the tensor components $\overset{2}{\vec{\Phi}}$, the tensor D_{ik} will approximately coincide with the tensor of small deformations ϵ_{ik}.

It will be seen from eq.(2.10) that the metric in the deformed medium is determined by the equations

$$G_{ik} = g_{ik} + 2D_{ik}. \tag{2.12}$$

Hence, from eqs.(I.5.9a) we may find the contravariant and mixed components of the metric tensor, and then the Christoffel symbols and the operation of absolute differentiation in the metric of the deformed medium.

3. Concluding Remarks /56

The reader has probably noted a certain arbitrariness in the construction of the tensor of small deformations and that of the tensor of finite deformations. We did in fact determine the increment of the displacement vector in the metric of the undeformed medium. It would have been possible, however, to use the metric of the deformed medium.

The arbitrariness in the choice of the metric is not fortuitous. This randomness is due to the fact that in a general study of the internal geometry of manifolds of coordinates x^i, the metric is introduced by definition and cannot be connected in advance with the properties of the manifold. These ideas are well known from modern differential geometry (Bibl.6). We have chosen the simplest method of defining the metric of a deformed medium and at the same time have defined the tensors of small and finite deformations. A different justification of the relations obtained is also possible. One could assert that the coordinates x^i in the undeformed medium simultaneously define two systems of coordinates, the Eulerian and Lagrangian. The expressions found for the tensor components of small and finite deformations are connected with the Eulerian coordinate system.

Section 3. Conditions of Compatibility

Equations (2.11) determine the finite-deformation tensor components if we know the components of the vector of displacement of an element of the continuous medium.

It is natural to pose the inverse problem; to find the displacement vector from the components of the finite-deformation tensor. This problem is solved by integrating a system of six nonlinear equations (2.11) with three unknown functions, the vector components u_i. Obviously, the possibility of a single-valued determination of the functions u_i from the system of equations (2.11) must be assured by satisfaction of additional conditions imposed on the components of the strain tensor. It is simplest here to start out from general geometrical considerations. The existence of the vector $\vec{u}$ is equivalent to the existence of the coordinate transformation formulas (1.2), and of transformation formulas inverse to eqs.(1.2), permitting us to pass from the metric in the deformed medium to the initial metric. But the initial metric is the

metric of Euclidean space. In this metric, the curvature tensor vanishes identically. Consequently, also in the deformed medium the curvature tensor will vanish if there exist the transformation formulas (1.2) or if there exists a displacement vector $\vec{u}$ as a single-valued function of the coordinates x^i at a fixed time t.

The fact that the components of the curvature tensor vanish is the wanted condition, which must be satisfied by the strain tensor components in order that the displacement vector determined from eqs.(2.11) be in existence. Making use of (I,10.7), we find: /57

$$\frac{1}{2}\left(\frac{\partial^2 G_{kj}}{\partial x^i \partial x^r} + \frac{\partial^2 G_{ir}}{\partial x^j \partial x^k} - \frac{\partial^2 G_{ij}}{\partial x^k \partial x^r} - \frac{\partial^2 G_{kr}}{\partial x^j \partial x^i}\right) +$$

$$+ G^{qs}\Gamma^{(D)}_{q,ir}\Gamma^{(D)}_{s,jk} - G^{qs}\Gamma^{(D)}_{q,ij}\Gamma^{(D)}_{s,kr} = 0. \tag{3.1}$$

where $\Gamma^{(D)}_{i,jk}$ are Christoffel symbols of the first kind expressed in terms of the metric tensor components of the deformed medium, G_{rs}. Substituting eqs.(2.11) into eqs.(3.1), we find the required compatibility conditions of eqs.(2.11), or the integrability conditions. A special case of eqs.(3.1), for small deformations, is given by the well-known Saint-Venant conditions*.

Section 4. Stress Tensor. Generalized Hooke's Law

1. Linear Generalization of Hooke's Law. Physical and Geometric Nonlinearity of the Equations of the Theory of Elasticity

The second tensor determining the state of the deformed medium is called the stress tensor. Its properties are well known from the principles of the mechanics of a continuous medium, and they will not be discussed here.

The stress tensor and the strain tensor are correlated by a system of relations resulting from the generalized Hooke's law. This connection is usually considered as linear and is accomplished by means of the elasticity tensor $C^{ik,rs}$. From energetic considerations it follows that the components of the elasticity tensor $C^{ik,rs}$ are symmetric in the labels i and k, r and s, and the pair of indices ik, rs. Thus in the most general case of anisotropy of the material, the tensor $C^{ik,rs}$ has only 21 independent components.

The generalized Hooke's law in the invariant form is expressed as follows:

$$\sigma^{ik} = C^{ik,rs} D_{rs}. \tag{4.1a}$$

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* Cf.E.Trefftz, Mathematical Theory of Elasticity and also (Bibl.7). ONTI, 1934

where σ^{ik} are the contravariant components of the stress tensor.

The expressions for the components of the deformation tensor in terms of the stress tensor are of the following form:

$$D_{rs} = \gamma_{rs,\,ik}\,\sigma^{ik}. \tag{4.1b}$$

where the quantities $\gamma_{rs,\,ik}$ are expressed in terms of $C^{ik,\,rs}$. To find these expressions, it is sufficient to perform the inversion of eqs.(4.1a), solving/58 the system of linear equations (4.1a) with respect to D_{rs}.

Let us consider an isotropic medium. In the case of an isotropic medium the elasticity tensor has only two substantially different components. All the components of the elasticity tensor can be expressed in terms of two independent quantities, which are constants in a homogeneous medium.

We now introduce the Lamé constants λ and μ:

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{4.2a}$$

The inverse relations are of the following form:

$$E = \frac{(3\lambda+2\mu)\,\mu}{\lambda+\mu}; \quad \nu = \frac{\lambda}{2(\lambda+\mu)}. \tag{4.2b}$$

In eqs.(4.2a) - (4.2b), E is Young's modulus, and ν is Poisson's constant. Equations (4.1a) can then be represented in the following form:

$$\sigma^{ik} = \lambda g^{ik}\,\theta + 2\mu g^{ir} g^{ks} D_{rs}. \tag{4.3}$$

where

$$\theta = g^{rs} D_{rs} \tag{4.4}$$

is the linear invariant of the strain tensor.

The quantities g^{pq} are the contravariant components of the metric tensor of the undeformed medium. The introduction of the metric tensor of the deformed medium would here be superfluous, since it would lead to a nonlinear relation between the components of the stress tensor and those of the strain tensor, which would be contradictory to eq.(4.1a).

We shall not dwell on the problem of justification of the analytic expression of the generalized Hooke's law defined by eqs.(4.1a) - (4.3), but shall adopt these equations as the direct consequences of experimental data, which are valid in a certain region of variation of the stress tensor and the strain tensor.

It follows from eqs.(4.3) - (4.4) that

$$\sigma^{ik} = (\lambda g^{ik} g^{rs} + 2\mu g^{ir} g^{ks}) D_{rs}. \tag{4.5a}$$

Passing to the covariant components of the stress tensor, we find:

$$\sigma_{ik} = (\lambda g_{ik} g^{rs} + 2\mu g_i^r g_k^s) D_{rs} \equiv \lambda g_{ik} g^{rs} D_{rs} + 2\mu D_{ik}. \tag{4.5b}$$

A comparison of eqs.(4.5a) and (4.1a) leads to the following expression for the components of the elastic tensor:

$$C^{ik,rs} = \lambda g^{ik} g^{rs} + 2\mu g^{ir} g^{ks}. \tag{4.6}$$

Equations (4.1a) - (4.1b) and (4.5a) express the linear Hooke's law, /59
since the components of the stress tensor and the strain tensor enter linearly into these relations. At the same time, it must be emphasized that the tensor D_{rs} contains nonlinear terms in the vector components $\vec{u}$ and their derivatives with respect to the coordinates x^i. In this connection, we distinguish between the physical nonlinearity and the geometrical nonlinearity of the equations of the elasticity theory or the equations of the mechanics of a continuous medium with properties more general than those of an elastic body*.

The nonlinear terms entering into the composition of the tensor of finite deformation determine the geometrical nonlinearity of the equations. Physical nonlinearity depends on the form of functional connection between the components of the stress tensor and those of the strain tensor.

2. The Nonlinear Hooke's Law

For an anisotropic body, on introduction of terms containing products and squares of the components of the strain tensor, we obtain

$$\sigma^{ik} = C_1^{ik,rs} D_{rs} + C_2^{ik,pq,rs} D_{pq} D_{rs}. \tag{4.7}$$

Here we meet two elastic tensors: the tensor $C_1^{ik,rs}$, discussed above, and

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* V.V.Novozhilov in his monograph (Bibl.11b) gives clear-cut definitions of these forms of nonlinearity.

the tensor of rank six $C_2^{ik,pk,rs}$. This tensor is symmetric in the labels i, k, p, q, r, s and the pairs of corresponding indices. An elementary calculation shows that this tensor has 79 substantially different components. Consequently, there are 100 substantially different components of the tensors $C^{ik,rs}$ and $C^{ik,pq,rs}$, taken together.

It is difficult to obtain the expressions determining the strain tensor components in terms of the stress tensor components by inversion of eqs.(4.7), since such inversion leads to the solution of a system of six quadratic equations, i.e., to the solution of an algebraic equation of twelfth degree. Such an equation in the general case cannot be solved in radicals.

All this indicates the great difficulties that arise in the study of problems of the mechanics of anisotropic elastic bodies with "physical nonlinearity".

Consider now an isotropic body. In order to set up the invariant expression of the generalized Hooke's law, including terms of the form $D_{pq}D_{rs}$, it is sufficient to consider the components of a tensor of rank six, constructed from the contravariant components of the metric tensor, permitting us, as a result of multiplication and contraction, to find additional linearly independent terms entering into the composition of the components of the stress tensor /60
σ^{ik}. These components of the required tensor of rank six, as can easily be verified, are expressed by three combinations: $g^{ik}g^{pq}g^{rs}$, $g^{pq}g^{ir}g^{ks}$, $g^{ip}g^{ik}g^{qs}$. Thus, the generalized nonlinear Hooke's law may be represented by the following invariant equation:

$$\sigma^{ik} = (\lambda g^{ik}g^{rs} + 2\mu g^{ir}g^{ks})D_{rs} + (c_1 g^{ik}g^{pq}g^{rs} + c_2 g^{pq}g^{ir}g^{ks} + c_3 g^{ip}g^{kr}g^{qs})D_{pq}D_{rs}. \tag{4.8}$$

Equation (4.8) contains five parameters determining the elastic properties of the medium: The Lamé constants λ and μ and the additional coefficients c_1, c_2, c_3. These coefficients are constants in a homogeneous body. All the above-enumerated coefficients are experimentally determined. Both eqs.(4.5a)-(4.6) and eq.(4.8) will contain contravariant components of the metric tensor of an undeformed medium. Indeed, the application of the metric tensor components of the deformed medium would lead, as is clear from eq.(2.12), to the introduction into eq.(4.8) of additional terms of the third dimension with respect to the components of the strain tensor. This would contradict eqs.(4.7) by which, in advance, we restricted the accuracy of the wanted relation. Equation (4.8)in essence coincides with the "Voigt-Murnaghan law". A critical analysis of certain consequences that result from relations analogous to eq.(4.8) is given elsewhere (Bibl.11b).

3. Concluding Remarks

In most works on the mechanics of deformable bodies, the construction of generalized formulations of Hooke's law is based on energetic considerations.

A detailed exposition of energetic principles would be outside the scope of this book, and we have therefore employed an outwardly formal method, postulating the invariance of the law sought and using the propositions of tensor algebra for the construction of its invariant formulation. The formalism of this method is illusory. It is well known from modern physics that the requirement of the invariance of the mathematical formulation of the laws of nature results from generalized principles, which are the expanded energetic considerations to which we have referred above.

Section 5. Equations of Motion of an Element of a Continuous Medium. The Linear Lamé Equations

1. Equations of Motion of an Element of a Continuous Medium in an Arbitrary System of Lagrange Coordinates

The equations to be considered determine the motion of an element of a deformed continuous medium. For this reason, in determining the components of the metric tensor, the Christoffel symbol, and the fundamental determinant, which are necessary for setting up the equations of motion, we must base our- /61
selves on eqs.(2.12). Here the components of the strain tensor enter into the fundamental determinant and the Christoffel symbols.

The fundamental determinant in the deformed medium will be called G. Various quantities connected with the deformed medium will be indicated by the index (D). The Christoffel symbols in the metric of the deformed medium will be indicated by brackets and braces:

$$\Gamma^{(D)}_{i,jk} = \left[{i \atop j\,k} \right]; \quad \Gamma^{i(D)}_{jk} = \left\{ {i \atop j\,k} \right\}. \tag{5.1}$$

The covariant derivative in the deformed medium will be indicated by $\nabla^{(D)}_j$.

In this notation the equations of motion of an element of a deformed medium and, in particular, of an element of an elastic or plastic body, are of the following form [cf., for instance (Bibl.7, 8, 11b)]:

$$\nabla^{(D)}_j \sigma^{ij} + \rho F^i = \rho \frac{\partial^2 u^i}{\partial t^2} \qquad (i,\ j = 1,\ 2). \tag{5.2a}$$

where ρ is the density of the material of the body, and F^i are the contravariant components of the mass forces. Making use of (I, 9.12), we represent eq.(5.2a) in the form

$$\partial_j \sigma^{ij} + \left\{ {i \atop j\,r} \right\} \sigma^{rj} + \left\{ {i \atop j\,r} \right\} \sigma^{ir} + \rho F^i = \rho \frac{\partial^2 u^i}{\partial t^2}. \tag{5.2b}$$

We now transform the sum of the Christoffel symbols $\left\{ {j \atop j\,r} \right\}$. On the basis of

eq.(I, 5.9a) and (I, 9.8), after the necessary changes in the dummy indices, we find

$$\left\{{}_{j\,r}^{\,j}\right\} = G^{js}\left[{}_{j\,r}^{\,s}\right] = \frac{1}{2} G^{js}\left(\frac{\partial G_{sj}}{\partial x^r} + \frac{\partial G_{sr}}{\partial x^j} - \frac{\partial G_{jr}}{\partial x^s}\right) = \frac{1}{2G}\frac{\partial G}{\partial G_{js}}\frac{\partial G_{js}}{\partial x^r},$$

Consequently,

$$\left\{{}_{j\,r}^{\,j}\right\} = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial x^r}. \tag{5.3}$$

Substituting eq.(5.3) into eq.(5.2a), we obtain

$$\frac{1}{\sqrt{G}}\partial_j(\sqrt{G}\,\sigma^{ij}) + \left\{{}_{j\,r}^{\,i}\right\}\sigma^{rj} + \rho F^i = \rho\frac{\partial^2 u^i}{\partial t^2}. \tag{5.4}$$

To the systems of equations (5.2a) or (5.4) we must associate the equation

$$\frac{\partial\rho}{\partial t} + \nabla_i^{(D)}(\rho v^i) = 0, \tag{a}$$

expressing the law of the conservation of mass. Equation (a) is usually called the equation of continuity [cf., for instance (Bibl.7)].

2. Linear Lamé Equations

We shall assume that the components of the tensor of the differential expansions $\overset{2}{\Phi}$ (Section 2) are small quantities. Accordingly, in the generalized linear Hooke's law (4.5a), we will replace the components of the finite-deformation tensor by the components of the tensor of small deformations ε_{ik}. We exclude the strain tensor components of the equations of motion (5.2a) from the components of the metric tensor and thus also from the Christoffel symbols. Then, the covariant derivative $\nabla_j^{(D)}$ is transformed into the covariant derivative ∇_j in the undeformed medium. Substituting the expressions for the strain tensor components (4.5a) into eq.(5.2a), we obtain, after simple transformations, the well-known Lamé equations in an arbitrary curvilinear coordinate system (Bibl.7):

$$\mu g^{ir}g^{ks}\nabla_k\nabla_s u_r + (\lambda+\mu)\,g^{ik}g^{rs}\nabla_k\nabla_r u_s + \rho F^i = \rho\frac{\partial^2 u^i}{\partial t^2}. \tag{5.5a}$$

Multiplying eq.(5.5a) by g_{ij}, performing the operation of contraction, and making use of the Ricci theorem, we obtain

$$\mu g^{ks}\nabla_k\nabla_s u_j + (\lambda+\mu) g^{ks}\nabla_j\nabla_k u_s + \rho F_j = \rho \frac{\partial^2 u_j}{\partial t^2}$$

$$(j,\ k,\ s=1,\ 2,\ 3). \qquad (5.5b)$$

Before setting up the nonlinear Lamé equations, let us consider several auxiliary propositions.

Section 6. Relationships between Covariant Derivatives in Deformed and Undeformed Media

1. Fundamental Determinant

Consider the expressions for the quantities determining the metric and parallel displacement in the space of the Lagrange coordinates of a deformed medium*, separating from these quantities the parts linearly connected with the strain tensor components.

Consider first the fundamental determinant. Applying the Taylor formula and eqs.(2.12), we find:

$$G = g + 2\frac{\partial g}{\partial g_{ik}} D_{ik} + \ldots = g(1 + 2g^{ik}D_{ik} + \ldots). \qquad (6.1)$$

2. Covariant and Contravariant Components of the Metric Tensor of a Deformed Medium /63

The covariant components of the metric tensor in the deformed medium are expressed by eqs.(2.12). Consider the contravariant components of the metric tensor. We introduce a system of generalized Kronecker delta δ^{ikj} which have the following properties: if the indices i, k, j form a positive cyclic permutation of the numbers 1, 2, 3, then the quantities δ_1^{ikj} will be equal to +1, whereas if the superscripts i, k, j form a negative cyclic permutation of the numbers 1, 2, 3, then the quantities δ^{ikj} are equal to -1; and if two identical numbers are present in the indices i,k,j, then the generalized Kronecker delta δ^{ikj} vanish**.

* For brevity, we shall speak hereafter of "in the deformed medium" etc..

** Detailed information on the properties of the quantities to which the just introduced generalized Kronecker delta δ^{ikj} belong, will be found in O.Veblen's book "Invariants of differential quadric forms", Chapter 1.,IL, 1948.

The fundamental determinant g may be represented by the following formula:

$$g = \delta^{ikj} g_{i1} g_{k2} g_{j3}. \tag{6.2a}$$

Similarly, bearing in mind eq.(6.1),

$$G = \delta^{ikj}(g_{i1} + 2D_{i1})(g_{k2} + 2D_{k2})(g_{j3} + 2D_{j3}) = g + 2gg^{ik}D_{ik} +$$
$$+ 4\delta^{ikj}(g_{i1}D_{k2}D_{j3} + g_{k2}D_{i1}D_{j3} + g_{j3}D_{i1}D_{k2}) + 8\delta^{ikj}D_{i1}D_{k2}D_{j3}. \tag{6.2b}$$

This equation may be simplified by making use of the identity

$$\frac{\partial^2 g}{\partial g_{ik}\partial g_{rs}} = \delta^{irp}\delta^{ksq}g_{pq}, \tag{a}$$

Then,

$$G = g + 2gg^{ik}D_{ik} + 2\delta^{irp}\delta^{ksq}g_{pq}D_{ik}D_{rs} + 8\delta^{ikj}D_{i1}D_{k2}D_{j3}; \tag{6.3}$$

Applying the Taylor formula and eqs.(2.12), we find

$$G^{ik} = g^{ik} + 2\frac{\partial g^{ik}}{\partial g_{rs}}D_{rs} + \dots \tag{b}$$

or, on the basis of (I, 5.9a) and eq.(6.2a),

$$G^{ik} = g^{ik} + 2\left(\frac{1}{g}\delta^{irp}\delta^{ksq}g_{pq} - g^{ik}g^{rs}\right)D_{rs} + \dots \tag{6.4}$$

The expressions in parentheses are the components of a tensor of rank four.

We will consider only that portion of this tensor which is symmetric in the labels r and s, since the tensor D_{rs} is symmetric in these indices.

Let us put /64

$$A^{ik,\,rs} = \frac{1}{g}\left(g^{irp}g^{ksq} + g^{isp}g^{krq}\right)g_{pq} - 2g^{ik}g^{rs}. \tag{6.5}$$

The tensor $A^{ik,rs}$ is symmetric in the superscripts i and k, r and s and the pairs of these indices. Thus, we find

$$G^{ik} = g^{ik} + A^{ik,\,rs}D_{rs} + \ldots \; . \tag{6.6a}$$

Hereafter we shall retain, in the equations of motion of the continuous medium, only terms which are quadratic in the tensor components $\overset{2}{\vec{\Phi}}$. Under this condition, the small-deformation tensor must be substituted for the finite-deformation tensor in eq.(6.6a). We obtain

$$G^{ik} = g^{ik} + A^{ik,\,rs}\varepsilon_{rs} + \ldots \quad . \tag{6.6b}$$

It is here assumed that the quantities ϵ_{rs} are nonvanishing. The case where any component of ϵ_{rs} vanishes requires special investigation (see also Sect. 8.4).

The differences

$$\overline{D}^{ik} = G^{ik} - g^{ik} = A^{ik,\,rs}D_{rs} + \ldots \tag{6.7}$$

may be regarded as components of the contravariant strain tensor. It is clear from eq.(6.6a) that, in the nonlinear theory of elasticity, we must distinguish the components of the contravariant strain tensor from the contravariant components of the covariant strain tensor defined by the equations

$$D^{ik} = g^{ir}g^{ks}D_{rs}. \tag{6.8}$$

We will not further go into these questions.

3. Christoffel Symbols in a Deformed Medium

Calculating the Christoffel symbols of the first kind in the metric of the deformed medium, we obtain

$$\left[{}^{\,j}_{ik}\right] = \Gamma_{j,\,ik} + 2\gamma_{j,\,ik}, \tag{6.9}$$

where

$$\gamma_{j,\,ik} = \frac{1}{2}\left(\frac{\partial D_{jl}}{\partial x^k} + \frac{\partial D_{jk}}{\partial x^l} - \frac{\partial D_{lk}}{\partial x^j}\right). \tag{6.10a}$$

Retaining in the equations of motion of an element of a continuous medium only terms that are quadratic in the tensor components $\overset{2}{\vec{\Phi}}$, instead of the quantity $\gamma_{j,ik}$ we must consider the quantities $\pi_{j,ik}$:

$$\pi_{j,ik}=\frac{1}{2}\left(\frac{\partial\varepsilon_{jl}}{\partial x^k}+\frac{\partial\varepsilon_{jk}}{\partial x^i}-\frac{\partial\varepsilon_{ik}}{\partial x^j}\right). \tag{6.10b}$$

On the basis of (I, 9.8) it is easy to establish that the quantities $\gamma_{j,ik}$ are Christoffel symbols of the first kind in a space whose metric is determined by the components of the strain tensor D_{rs}. The quantities $\pi_{j,ik}$ are Christoffel symbols of the first kind in a space with the metric tensor ε_{ik}. /65

Let us find the Christoffel symbols of the second kind in the metric of the deformed medium. We have

$$\left\{{j \atop i\,k}\right\}=G^{js}\left[{s \atop i\,k}\right]=(g^{js}+A^{js,pr}D_{pr}+\ldots)(\Gamma_{s,ik}+2\gamma_{s,ik}). \tag{a}$$

Retaining in the left side of eq.(a) the terms linearly dependent on the strain tensor components and their derivatives, we find

$$\left\{{j \atop i\,k}\right\}=\Gamma^j_{ik}+2g^{js}\gamma_{s,ik}+A^{js,pr}\Gamma_{s,ik}D_{pr}+\ldots\ . \tag{6.11}$$

We now introduce the notation

$$P_{ik\cdot}^{\cdot\cdot j}=2g^{js}\gamma_{s,ik}+A^{js,pr}\Gamma_{s,ik}D_{pr}. \tag{6.12a}$$

Since we shall retain in our equations only nonlinear terms which are quadratic in tensor components $\overset{2}{\vec{\Phi}}$, we replace the tensor $\overset{3}{\vec{P}}$ by the tensor $\overset{3}{\vec{N}}$ with components expressed as follows:

$$N_{ik\cdot}^{\cdot\cdot j}=2g^{js}\pi_{s,ik}+A^{js,pr}\Gamma_{s,ik}\varepsilon_{pr}. \tag{6.12b}$$

Thus,

$$\left\{{j \atop i\,k}\right\}=\Gamma^j_{ik}+P_{ik\cdot}^{\cdot\cdot j}, \tag{6.13a}$$

or, in accordance with eq.(6.12b),

$$\left\{{j \atop i\,k}\right\}=\Gamma^j_{ik}+N_{ik\cdot}^{\cdot\cdot j} \tag{6.13b}$$

We shall show that the quantities $P_{ik\cdot}^{\cdot\cdot j}$ are mixed components of a tensor of third rank. This proof is also extended to the quantities $N_{ik\cdot}^{\cdot\cdot j}$.

Now, setting up the transformation formulas for the Christoffel symbols $\{{}_{i}{}^{j}{}_{k}\}$ and Γ^{j}_{ik} by eqs.(I, 9.7b), we find that their difference $P_{ik}^{\cdot\cdot j}$ (or, approximately, $N_{ik}^{\cdot\cdot j}$) obeys the transformation formulas for the mixed components of a tensor of third rank. This tensor is symmetric in the indices i and k.

4. Covariant Derivative in a Deformed Medium

Consider again (I, 9.12). We have

$$\nabla_s^{(D)} T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} = \partial_s T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} + \{{}_{r}{}^{i}{}_{s}\}\, T^{rk\cdot\cdot}_{\cdot\cdot j\cdot} + \{{}_{r}{}^{k}{}_{s}\}\, T^{ir\cdot\cdot}_{\cdot\cdot j\cdot} - \{{}_{j}{}^{r}{}_{s}\}\, T^{ik\cdot\cdot}_{\cdot\cdot r\cdot} + \dots \quad . \tag{6.14}$$

On the basis of eq.(6.13a), we obtain from eq.(6.14):

$$\nabla_s^{(D)} T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} = \nabla_s T^{ik\cdot\cdot}_{\cdot\cdot j\cdot} + P_{rs\cdot}^{\cdot\cdot i}\, T^{rk\cdot\cdot}_{\cdot\cdot j\cdot} + P_{rs\cdot}^{\cdot\cdot k}\, T^{ir\cdot\cdot}_{\cdot\cdot j\cdot} - P_{js\cdot}^{\cdot\cdot r}\, T^{ik\cdot\cdot}_{\cdot\cdot r\cdot} + \dots \quad . \tag{6.15a}$$

In particular, we find for the contravariant vector /66

$$\nabla_s^{(D)} a^i = \nabla_s a^i + P_{ks\cdot}^{\cdot\cdot i} a^k \tag{6.15b}$$

and for the covariant vector

$$\nabla_s^{(D)} a_i = \nabla_s a_i - P_{is\cdot}^{\cdot\cdot k} a_k. \tag{6.15c}$$

Let us consider the commutativity of the operators $\nabla_s^{(D)}$ and ∇_r. From (I, 10.9) and eq.(6.15b), we find

$$\nabla_s^{(D)} \nabla_r a^i - \nabla_r \nabla_s^{(D)} a^i = P_{rs\cdot}^{\cdot\cdot j} \nabla_j a^i + a^j \nabla_r P_{js\cdot}^{\cdot\cdot i} + R_{sr,j\cdot}^{\cdot\cdot\cdot i} a^j. \tag{6.16}$$

Equation (6.16) is simplified if the space filled by the medium is Euclidean. It is precisely this case that we shall consider hereafter. But even in this case, the operators ∇_r and $\nabla_r^{(D)}$ are noncommutative, which considerably complicates the nonlinear equations of motion of an element of an elastic or plastic body.

5. Conclusion

The relations (6.1) - (6.16) found by us permit deriving approximate nonlinear equations of motion of an element of an elastic or plastic body in the metric of the undeformed medium. These relations permit an inversion: all the quantities required for this construction can be expressed in the metric of the deformed medium.

To summarize, we may assert that we have constructed a fundamental system of quantities which permit us to set up the nonlinear approximate equations of motion of an element of a continuous medium in an arbitrary curvilinear system

of coordinates, i.e., in the invariant form, in one of two metrics: either in the metric of the undeformed medium or in the metric of the deformed medium.

Section 7. Nonlinear Lamé Equations*

In order to set up a system of equations permitting investigation of the motion of the particles of elastic bodies under finite deformations, we must make use of the equations of motion, eqs.(5.4), and the nonlinear Hooke's law (Murnaghan's law) expressed by eqs.(4.7) - (4.8). Here we must know the components of the elastic tensor. Since the components of the elastic tensor $\overset{6}{\vec{C}}_2$ have been little investigated, even in the case of an isotropic body, we shall confine ourselves to the linear Hooke's law (4.5a).

We recall that eqs.(4.5a) contain terms nonlinear in the tensor components $\overset{2}{\vec{\Phi}}$ so that, without considering physical nonlinearity, we shall preserve geometrical nonlinearity. /67

Let us bear in mind eqs.(6.1) and (6.13b). We represent eq.(2.11) in the following form:

$$D_{ik} = \varepsilon_{ik} + \frac{1}{2}\nabla_i u^r \nabla_k u_r. \tag{7.1}$$

We shall denote by σ_0^{ik} the stress tensor components expressed on the basis of eqs.(4.5a) in terms of the components of the tensor of small deformations ε_{rs}. Then,

$$\sigma^{ik} = \sigma_0^{ik} + \frac{1}{2}(\lambda g^{ik}g^{rs} + 2\mu g^{ir}g^{ks})\nabla_r u^j \nabla_s u_j. \tag{7.2}$$

The equations of motion of an element of a continuous medium [eqs.(5.4)] may be represented in the form

$$\frac{1}{\sqrt{g}}\partial_j(\sqrt{g}\,\sigma_0^{ij}) + \Gamma^i_{jr}\sigma_0^{jr} + \rho F^i + \Phi^i = \rho\frac{\partial^2 u^i}{\partial t^2}. \tag{7.3}$$

where

$$\Phi^i = \sigma_0^{ij}\partial_j\theta + N_{jr\cdot}^{\cdot\cdot i}\sigma_0^{jr} + \frac{1}{2\sqrt{g}}\partial_j[\sqrt{g}(\lambda g^{ij}g^{rs} +$$

$$+ 2\mu g^{ir}g^{js})\nabla_r u^k \nabla_s u_k] + \frac{1}{2}\Gamma^i_{jr}(\lambda g^{rj}g^{pq} + 2\mu g^{rp}g^{jq})\nabla_p u^k \nabla_q u_k. \tag{7.4}$$

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* These equations were first considered by us in Reference 23b, Part II, Section 4.

The quantities Φ^i determine the influence of geometrical nonlinearity on the motion of an element of an elastic body. These quantities are equivalent to additional body forces.

Thus, to set up the nonlinear Lamé equations, it is sufficient to introduce the additional body forces Φ^i into the left side of the linear Lamé equations (5.5a) or (5.5b). We obtain

$$\mu g^{ir} g^{ks} \nabla_k \nabla_s u_r + (\lambda + \mu) g^{ik} g^{rs} \nabla_k \nabla_r u_s + \rho F^i + \Phi^i = \rho \frac{\partial^2 u^i}{\partial t^2}, \tag{7.5a}$$

$$\mu g^{ks} \nabla_k \nabla_s u_i + (\lambda + \mu) g^{ks} \nabla_j \nabla_k u_s + \rho F_j + \Phi_j = \rho \frac{\partial^2 u_j}{\partial t^2}. \tag{7.5b}$$

These invariant equations may obviously be represented in vector form:

$$\mu \nabla^2 \vec{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \vec{u} + \rho \vec{F} + \vec{\Phi} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}. \tag{7.6}$$

where

$$\nabla^2 = g^{kr} \nabla_k \nabla_r \tag{7.7}$$

is the Laplace operator in an arbitrary curvilinear system of coordinates, and

$$\operatorname{div} \vec{u} = \nabla_r u^r = g^{rs} \nabla_r u_s. \tag{7.8}$$

Section 8. Initial and Boundary Nonlinear Conditions. Conditions of Contact of Layers

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1. Initial Conditions

The statement of problems of the mechanics of deformable bodies includes, as a necessary element, the assignment of a system of initial and boundary conditions. Since the differential equations of motion of an element of a continuous medium are equations of the second order with respect to the time t, the classical initial conditions become applicable: At some time $t = t_0$, arbitrarily called the initial time, the positions and velocities of the elements of the deformable medium must be assigned. In connection with the fact that the positions and velocities of the elements of the medium are expressed in terms of the displacement vector components and their time derivatives, we have the following initial conditions:

$$u_i(t_0, x^i) = u_{i0}(x^i), \tag{8.1a}$$

$$\dot{u}_i(t_0, x^i) = \dot{u}_{i0}(x^i). \tag{8.1b}$$

where the dot denotes differentiation with respect to time.

2. Nonlinear Boundary Conditions

We shall consider below only the case where the continuous medium is a solid body and, in particular, an elastic body.

A feature of boundary problems of the mechanics of solid deformable bodies under finite displacements and deformations of their elements is the assignment of boundary conditions on the deformed surface of the body whose shape is to be determined. On the deformed surface of the body may be assigned: a) the components of the displacement vector, b) the components of the stress vector, and c) the mixed boundary conditions. The expressions of the boundary conditions, like the equations of motion of an element of a continuous medium considered above, result from the general equation of dynamics. The derivation of these conditions will not be discussed here, and the reader is referred to general Handbooks on elasticity theory (cf., for example, Bibl.11b).

We will confine ourselves here to more elementary considerations. Assume that in the case of the boundary problem a), the displacement vector components are assigned as functions of the Lagrangian coordinates of the points of the body surface. Problem b) will be discussed in greater detail.

The components of the stress vector $\vec{f}$ are expressed by the equations

$$\sigma^{ik} n_k = f^i \tag{8.2a}$$

or

$$\sigma_{ik} n^k = f_i \qquad (i,\ k = 1,\ 2,\ 3). \tag{8.2b}$$

where n_k are the covariant components of the unit vector of an external normal $\vec{n}$ to the deformed surface of the body. /69

Since the shape of the deformed surface of the body is to be determined, let us express the components of the unit vector of the external normal $\vec{n}$ in terms of the components of the unit vector of the external normal $\vec{n}_0$ to the undeformed surface. Let us make use of eq.(1.1) and assume that the equations of the surface of the body in parametric form read as follows:

$$x^i = x^i(\xi^1,\ \xi^2) \qquad (i = 1,\ 2,\ 3). \tag{8.3}$$

where ξ^α $(\alpha = 1,2)$ are the Gaussian coordinates of the points of the surface of the body.

We note that eqs.(8.3) remain unchanged under deformation of the body.

The coordinate vectors of the local coordinate basis on the surface of the body are determined by the equations

$$\vec{e}_\alpha = \frac{\partial \vec{r}}{\partial \xi^\alpha} = \frac{\partial \vec{r}}{\partial x^i}\frac{\partial x^i}{\partial \xi^\alpha} = \vec{e}_i \frac{\partial x^i}{\partial \xi^\alpha} \qquad (i=1,\ 2,\ 3;\ \alpha=1,\ 2). \tag{8.4}$$

The unit vector of the external normal to the deformed surface of the body is determined by (I, 3.1):

$$\vec{n} = \frac{\vec{e}_1 \times \vec{e}_2}{|\vec{e}_1 \times \vec{e}_2|}. \tag{8.5}$$

It is here assumed that the choice of the parameters ξ^α will ensure the direction selected for the unit vector $\vec{n}$. It follows from eqs.(8.4) and (8.5) that

$$\vec{n} = \frac{\vec{e}_i \times \vec{e}_k \dfrac{\partial x^i}{\partial \xi^1}\dfrac{\partial x^k}{\partial \xi^2}}{\left|\vec{e}_i \times \vec{e}_k \dfrac{\partial x^i}{\partial \xi^1}\dfrac{\partial x^k}{\partial \xi^2}\right|}. \tag{8.6}$$

Making use of eq.(I, 8.6), we find

$$(\vec{e}_i \times \vec{e}_k)_r = \sqrt{G}\,(\delta_i^p \delta_k^q - \delta_i^q \delta_k^p) = \sqrt{G}\,\delta_{ikr}. \tag{8.7}$$

where δ_{ikr} are quantities analogous to those considered in Sect.6.2. These quantities are equal to +1 if the labels i, k, r form a positive cyclic permutation of the numbers 1, 2, 3, while they are equal to -1 if this permutation is negative and vanish in all other cases. Then, the covariant components of the vector $\vec{n}$, determined by eq.(8.6), will be expressed as follows: /70

$$n_j = \frac{C_j}{\sqrt{B_{rs}G^{rs}}}; \tag{8.8}$$

where

$$C_j = \delta_{ikj}\frac{\partial x^i}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^2}, \tag{8.9a}$$

$$B_{rs} = \delta_{ikr}\delta_{pqs} \frac{\partial x^i}{\partial \xi^1} \frac{\partial x^k}{\partial \xi^2} \frac{\partial x^p}{\partial \xi^1} \frac{\partial x^q}{\partial \xi^2}, \tag{8.9b}$$

and where the quantities C_j and B_{rs} do not depend on the deformation of the body.

The contravariant components of the metric tensor G^{rs} are determined from eqs.(6.6a). Making use of these equations, we get

$$\begin{aligned}(B_{rs}G^{rs})^{-\frac{1}{2}} &= [B_{rs}(g^{rs} + A^{rs,pq} D_{pq} + \ldots)]^{-\frac{1}{2}} = \\ &= (B_{rs}g^{rs})^{-\frac{1}{2}} \left[1 - \frac{B^{rs}A^{rs,pq} D_{pq} + \ldots}{B_{rs}g^{rs}}\right]^{-\frac{1}{2}}.\end{aligned} \tag{8.10a}$$

Confining ourselves to the linear approximation, we find

$$n_j = C_j (B_{rs}g^{rs})^{-\frac{1}{2}} \left[1 - \frac{A^{rs,pq} B_{rs} \varepsilon_{pq}}{2B_{rs} g^{rs}} + \ldots\right]. \tag{8.10b}$$

Noting that the covariant components of the unit vector $\vec{n}_0$ are expressed by the equations

$$n_{0j} = C_j (B_{rs} g^{rs})^{-\frac{1}{2}}, \tag{8.11}$$

we obtain

$$n_j = n_{0j} - \frac{A^{rs,pq} B_{rs} \varepsilon_{pq}}{2B_{rs}g^{rs}} n_{0j} + \ldots \quad . \tag{8.12}$$

On the basis of eq.(8.12), the condition (8.2a) for the deformed surface of the body takes the following form:

$$\left(1 - \frac{A^{rs,pq} B_{rs} \varepsilon_{pq}}{2B_{rs}g^{rs}} + \ldots\right) \sigma^{ij} n_{0j} = f^i. \tag{8.13}$$

Equations (8.13) approximately express the nonlinear boundary conditions of the problem b). After these remarks, the analytical statement of problem c)

is now obvious.

3. Conditions of Contact on Surfaces of Separation between Media with Matter of Different Mechanical Characteristics

In the shell theory we have to do with layered aggregates. The theory of layered shells is developed in a monograph (Bibl.1). The conditions for the interfaces between layers can be very varied and depend on the method of con- /71 struction of the layered shell which is essentially a system of shells.

Assume, for instance, that the design of a layered shell ensures the continuity of the field of displacements. The stress tensor components on the interface between the layers must obey the conditions resulting from Newton's Third Law. Thus, on the interface of media labeled k and k + 1, the conditions

$$\overset{(k)}{u_i}=\overset{(k+1)}{u_i},\quad \overset{(k)}{f^i}=\overset{(k+1)}{f^i}\qquad (i=1,\ 2,\ 3). \tag{8.14}$$

must be satisfied.

4. General Characterization of the Formulation of Nonlinear Problems of the Theory of Elasticity

The problems of the nonlinear theory of elasticity belong to two classes. The first class consists of weakly nonlinear problems and the second, of strongly nonlinear problems. The problems of the first class are characterized by the fact that the absolute values of the components of the tensor $\overset{2}{\vec{\Phi}}$ are proper fractions. This permits us to neglect, in the equations of motion of the elements of an elastic body, the nonlinear terms with an index of homogeneity greater than two, relative to the components of the tensor $\overset{2}{\Phi}$.

All the remaining cases are strongly nonlinear. However, this classification is arbitrary and in certain cases inapplicable. We will discuss these cases later in the text.

The relations of the above nonlinear theory relate to weakly nonlinear problems. The question naturally arises as to the limits of applicability of weakly nonlinear and strongly nonlinear theory. These limits obviously depend on two groups of factors.

The first group of factors limits the applicability of the equations of the theory of elasticity by the physical properties of the material: The deformation must be so small that Hooke's law (4.5a) or the Voigt-Murnaghan law (4.8) are satisfied. These considerations emphasize the desirability of making use of the equations of the weakly nonlinear theory and narrow the limits of application of the strongly nonlinear theory.

The second group of factors has a kinematic meaning and restricts the applicability of the equations of the weakly nonlinear theory or, more exactly, forces us to verify their applicability in solving specific problems. In fact, these equations were formally set up on the basis of expansions according to

the degree of homogeneity of the terms containing components of the tensor ${}^{2}\vec{\Phi}$. In the equations were retained only the terms which, on application of Hooke's law (4.5a), will contain terms linear in the tensor components ${}^{2}\vec{\Phi}$ or their derivatives and squares. /72

A simple example will show that this procedure does not always lead to success. Let us consider the equation which is familiar from courses on the strength of materials:

$$\frac{EIy''}{(1+y'^2)^{\frac{3}{2}}}=M. \qquad (a)$$

Applying the above procedure, we find

$$EJy''=M, \qquad (b)$$

i.e., here we might reach the erroneous conclusion that the equations were weakly nonlinear and coincided with the linear theory.

These errors might have been avoided if we had noted that y'^2 is a component of the strain tensor ${}^{2}\vec{D}$ in which, in this case, only the term containing y'^2 does not vanish.

Thus, the expansion considered by us may be simplified by replacing the tensor D_{ik} by the tensor ϵ_{ik}, if the components of ϵ_{ik} are nonvanishing. If some tensor components ϵ_{ik} in some specific problem do vanish, then, in the corresponding tensor components D_{ik}, we must retain the terms of higher order and eliminate them from the equations only after an additional analysis making allowance for the special features of the problem.

All above statements lead to the conclusion that it is expedient separately to consider the quantities characterizing nonlinear deformations "as a whole"*.

Section 9. Internal and External Nonlinear Problems

The mechanics of deformable bodies comprises two fundamental problems which we shall call "internal" and "external".

The internal problem is to determine the stressed and strained states of elements of a moving body. The external problem is to describe the motion of the set of elements of a body or of the body "as a whole" relative to a system of Eulerian coordinates.

The internal state of the elements of a body is determined by the stress,

* Cf.(Bibl.11b) and also I.Gekkeler, Statics of an Elastic Body, ONTI, 1934, pp.80-96.

strain, and elastic tensors. It is natural to apply here the Lagrangian coordinates. The external problem is solved after determination of the displacement vector $\vec{u}$, which on the basis of eqs.(1.1) - (1.2) permits establishing the configuration of the deformed body at arbitrary time.

In V.V.Novozhilov's monograph (Bibl.11b), four groups of problems of nonlinear elasticity theory are defined. In accordance with his conclusions, we will hereafter focus our attention on problems physically linear but geometrically nonlinear, since it is precisely this group of problems that is closest to the nonlinear problems of the theory of elastic shells*. /73

The division of the general problem of the mechanics of elastic deformable bodies into an external and an internal problem was quite fully accomplished by Kirchhoff and Clebsch in the statics of thin rods. They found that the internal problem of the statics of thin rods is linear.

The stressed and strained state of the elements of a thin rod was described, perhaps in first approximation but with sufficient accuracy, by the solution of the well-known Saint-Venant problem with indeterminate parameters, depending on the solution of the external problem. The solution of the external problem required the integration of systems of nonlinear differential equations analogous to those known from the dynamics of a solid body.

The above-given equations of nonlinear elasticity theory do not permit a complete separation of the external and internal problems, although the use of the coordinates x^i instead of Eulerian Cartesian coordinates makes it possible to advance considerably in this direction.

Considering the quantities entering into the equation of the theory of elasticity, we note that the internal deformed state of the elements of a body must, according to Kirchhoff and Clebsch, be described by the tensor of small deformations ϵ_{ik}. Among the quantities defining the state of the body "as a whole", the components of the antisymmetric tensor Ω_{ik}, expressed by eqs.(2.7) and the quantities $\{{}^{i}_{jk}\}$ must be included.

As noted in (I, Sect.9), the Christoffel symbols $\{{}^{i}_{jk}\}$ have a nature similar to that of the tensor of instantaneous angular velocity of a solid body or to that of the vector of angular velocity of rotation of a natural trihedron, considered in the theory of thin rods. We recall that this vector, according to the Kirchhoff-Clebsch theory, satisfies equations of equilibrium analogous to the dynamic Euler equation determining the motion of a rigid body about a fixed point.

Turning to the equations of nonlinear elasticity theory, we note that the difficulties arising in the general solution of the question of subdividing the problem of dynamic deformation of an elastic body into an external problem and an internal problem consists in analytically expressing a generalized Hooke's law, relating the stress tensor components to the components of the finite-deformation tensor and to the analytic expressions for the latter. It follows

* Cf. (Bibl.11b, pp.125-126).

from eqs.(2.6), (2.7), and (2.11) that the tensor components D_{ik} can be represented in the following form: /74

$$D_{ik}=\varepsilon_{ik}+\frac{1}{2}g^{rj}[\varepsilon_{ir}\varepsilon_{kj}+\varepsilon_{ir}\Omega_{kj}+\varepsilon_{kj}\Omega_{ir}+\Omega_{ir}\Omega_{kj}]. \tag{9.1}$$

If, in accordance with V.V.Novozhilov, we consider the case of quantities ϵ_{ik} which are small in comparison with unity, then we obtain approximately*

$$D_{ik}\cong\varepsilon_{ik}+\frac{1}{2}g^{rj}\Omega_{ir}\Omega_{kj}. \tag{9.2}$$

It will be clear from eqs.(9.1) and (9.2) that D_{ik} and, consequently, the components of the stress tensor σ^{ik} contain quantities relating to both the external and internal problems.

Considering the equations of the nonlinear theory of elasticity, we may state that only the conditions of compatibility (3.1) belong exclusively to the external problem.

There is a resemblance between conditions (3.1) and the equations of equilibrium of thin rods. This resides in the fact that the derivatives of the Christoffel symbols, entering into the conditions (3.1), are analogous to the derivatives of the instantaneous angular velocity components of a natural trihedron of the axis of the rod, which enter into the equations of equilibrium of thin rods. The difference is that the conditions of compatibility do not have a kinetic but only a kinematic meaning.

Section 10. Extension of the Kinematic Relations of the Kirchhoff-Clebsch Thin-Rod Theory to Shell Theory

The theory of thin shells proposed by Kirchhoff and Clebsch is based on the kinematic relations referring to simplified assumptions related with certain concepts on the deformation of beams.

We will show that the kinematic relations of the theory of thin rods can be generalized into the three-dimensional problems of the theory of elasticity, and first of all into the problems of the theory of shells**.

Let us consider, in the deformed body, the point $M(x_0^i)$ and the local coordinate basis associated with it. Let us superpose on this basis the axis of a

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* Equation (9.2) corresponds to formulas (1, 111) of the author's book (Bibl.11b).

** We have been guided by the exposition of the Kirchhoff-Clebsch theory in Gekkeler's book "Statics of an Elastic Body", ONTI, 1934, Section 30a*, using it in the generalization of the analytic apparatus of tensor analysis.

fixed Cartesian (Eulerian) system of coordinates y_0^i. The neighborhood of the point M is thus determined by the coordinates y_0^i which are functions of the coordinates x^i. The coordinates y_0^i vanish at point M.

Let $\vec{r}(x_0^i, y_0^i)$ be a radius vector drawn from point $M(x_0^i)$ to point $N(x_0^i, y_0^i)$. Let $\vec{v}$ be the displacement of point N relative to point M: /75

$$\vec{v}(t,\ x_0^i;\ y_0^i)=\vec{u}(t,\ x_0^i;\ y_0^i)-\vec{u}(t,\ x_0^i;\ 0). \tag{a}$$

Clearly,

$$\vec{v}(t,\ x_0^i;\ 0)=0;\quad \frac{\partial\vec{v}(t,\ x_0^i;\ 0)}{\partial x_0^k}=0. \tag{b}$$

It goes without saying that eq.(a) has a meaning in a curvilinear coordinate system only when the vector $\vec{u}(t, x_0^i, y_0^i)$ undergoes displacement parallel, in the sense of Levi-Civita, to point M.

We introduce the vector

$$\vec{R}(t,\ x_0^i;\ y_0^i)=\vec{r}(x_0^i;\ y_0^i)+\vec{v}(t,\ x_0^i;\ y_0^i) \tag{10.1}$$

and investigate the variation of this vector relative to a moving coordinate base, with the origin being displaced along the coordinate line x^k.

A network of the local coordinate system y^i is associated with the moving base. When the base moves through the points M and N, which are fixed in space, the points of this network will be continuously displaced, so that the points M and N are in motion relative to the system of coordinates y^i with its origin at the fixed point $M'(x^i)$.

Let us assume, for definiteness, that the motion of the coordinate base is determined by the relations

$$x^i=x_0^i;\quad y^i=y_0^i;\quad x^k=x_0^k+s^k;\quad y^k=y_0^k-s^k. \tag{10.2}$$

where the index k is fixed. The parameter s^k determines the motion of the coordinate base.

Consider the vector $\vec{v}$. Its components in the moving system of coordinates y^j will be

$$v^j=v^j(t,\ x^i\ x^k;\ y^i,\ y^k). \tag{10.3}$$

As for the vector $\vec{r}$, representing its decomposition into the components $\overrightarrow{MM'}$ and $\overline{M'N}$, we determine its components by the equation

$$r^j \overset{*}{=} y^j + \delta^j_k s^k = y^j_0. \tag{10.4}$$

The sign $\overset{*}{=}$ indicates invariance of eq.(10.4). We emphasize that eq.(10.4) must be regarded as the definition of vector r, not subject to proof, but corresponding instead to elementary geometrical concepts. /76

We revert to the vector $\vec{R}$. It follows from the definition of this vector that its absolute derivative with respect to the variable s^k vanishes*:

$$\frac{\partial R^i}{ds^k} + \{^{\ i}_{j\ k}\}_M R^j = 0. \tag{10.5}$$

Further, we find

$$\frac{\partial v^i}{\partial s^k} = \frac{\partial v^i}{\partial x^k} - \frac{\partial v^i}{\partial y^k}; \quad \frac{\partial r^i}{\partial s^k} = 0, \tag{10.6}$$

Consequently,

$$\frac{\partial v^i}{\partial x^k} - \frac{\partial v^i}{\partial y^k} + \{^{\ i}_{j\ k}\}_M (y^j_0 + v^j) = 0. \tag{10.7a}$$

Noting that it follows from eqs.(10.2), (10.3) that

$$\frac{\partial v^i}{\partial x^k} = \frac{\partial v^i}{\partial x^k_0}; \quad \frac{\partial v^i}{\partial y^k} = \frac{\partial v^i}{\partial y^k_0}, \tag{c}$$

we find finally

$$\frac{\partial v^i}{\partial x^k_0} - \frac{\partial v^i}{\partial y^k_0} + \{^{\ i}_{j\ k}\}_M (y^j_0 + v^j) = 0. \tag{10.7b}$$

These relations are an extension of the Kirchhoff-Clebsch kinematic equations to the three-dimensional problems of elasticity theory. The Christoffel symbols $\{^{\ i}_{j\ k}\}$, as already noted, are in this case associated with the known

* We recall that the point of attachment of the vector $\vec{R}$ is the point M.

quantities p, q, r of the theory of thin rods*.

According to the Kirchhoff-Clebsch theory, the stressed state of the elements of a rod is determined not by the vector $\vec{u}$ of absolute displacement of an element of the rod but by the vector of relative displacement $\vec{v}$. The "internal" problem is solved in the components of the vector $\vec{v}$.

Assume that the stressed state of an element of a shell is also determined by the vector $\vec{v}$. We then represent eq.(10.7b) in the following form:

$$\frac{\partial v^i}{\partial y_0^k} = \nabla_k^{(D)} v^i + \left\{ {}^{\,i}_{j\,k} \right\}_M y_0^j. \tag{10.8}$$

We also assume that the base area of a shell can be chosen such that the quantities $\nabla_k^{(D)} v^i$ shall be small. Then, we find in first approximation /77

$$\frac{\partial v^i}{\partial y_0^k} \cong \left\{ {}^{\,i}_{j\,k} \right\}_M y_0^j. \tag{10.9}$$

These relations permit us to derive the vector components v^i and thus to solve the internal problem. The external problem is solved by applying the equations of motion.

Other applications of eqs.(10.8) are also possible. For example, it follows from this equation that

$$\nabla_k^{(D)} u^i(t,\ x_0^i;\ y_0^i) = \nabla_k^{(D)} u^i(t,\ x_0^i:\ 0) + \frac{\partial v^i}{\partial y_0^k} + \left\{ {}^{\,i}_{j\,k} \right\}_M y_0^j. \tag{10.10}$$

If, as a result of the smallness of ∂v^i, we neglect the term $P_{k\cdot j}^{\cdot\,\cdot\, i} v^j$, then eq.(10.10) takes the form

$$\nabla_k u^i(t,\ x_0^i;\ y_0^i) = \nabla_k u^i(t,\ x_0^i:\ 0) + \frac{\partial v^i}{\partial y_0^k} + \left\{ {}^{\,i}_{j\,k} \right\}_M y_0^j \tag{10.11}$$

Equations (10.10) and (10.11) permit us to develop a method of application of three-dimensional problems of the theory of elasticity to two-dimensional problems, different from the methods known at present.

If we choose the base area such that the covariant derivatives such as

* Compare with pp.86, 87 of the above-cited book by I.Gekkeler "Statics of an Elastic Body", ONTI, 1934.

$\nabla_k^{(D)} u^i(t, x_0^i; 0)$ or $\nabla_k u^i(t, x_0^i; 0)$ are sufficiently small in absolute value and the y^i are also sufficiently small, then eqs.(10.10) - (10.11) permit a linearization of the expression for the tensor components D_{ik} and a linearization of the above-derived equations for the nonlinear theory of elasticity.

Section 11. Potential Energy of Deformation and Kinetic Energy of the Elastic Body

Without dwelling on the well-known conditions of the existence of potential deformation energy as a function of the strain-tensor components, we wish to state that, in adopting for phenomenological considerations the Hooke-Voigt-Murnaghan law in the form of eq.(4.8) or in the more general form of eq.(4.7), we implicitly assumed that the above-mentioned conditions of existence were satisfied*.

The elementary work of deformation has the following form:

$$\delta A = \iiint\limits_{(V)} \sigma^{ik}\, \delta_i D_k\, dV. \tag{11.1}$$

where V is the volume of the deformed body. The Pfaff form $\sigma^{ik}\,\delta D_{ik}$ is integrable if eqs.(4.7) are satisfied. The conditions of integrability are satisfied by the symmetry properties of the elastic tensors $C_1^{ik,rs}$ and $C_2^{ik,pq,rs}$, indicated in Sect.4.2. /78

Hereafter we shall make use of the linear Hooke's law. Integrating in this case the Pfaff form $\sigma^{ik}\,\delta D_{ik}$, we find

$$A = \frac{1}{2} \iiint\limits_{(V)} C_1^{ik,\,rs} D_{ik} D_{rs}\, dV = \frac{1}{2} \iiint\limits_{(V)} \sigma^{ik} D_{ik}\, dV. \tag{11.2a}$$

In the more general case,

$$A = \frac{1}{2} \iiint\limits_{(V)} C_1^{ik,\,rs} D_{ik} D_{rs}\, dV + \frac{1}{6} \iiint\limits_{(V)} C_2^{ik,\,pq,\,rs} D_{ik} D_{pq} D_{rs}\, dV, \tag{11.2b}$$

and here

$$A \neq \frac{1}{2} \iiint\limits_{(V)} \sigma^{ik} D_{ik}\, dV.$$

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* More details on the conditions of existence of potential deformation energy as a function of the components of the tensor D_{ik} will be found in A.Love's book "Theory of Elasticity", and also in (Bibl.11b).

Equations (11.2a) - (11.2b) determine the potential energy of deformation A. The resultant expressions are invariant under point transformations of the coordinates and determine A in an arbitrary curvilinear coordinate system.

The kinetic energy of an elastic body is determined by the following equation:

$$T=\frac{1}{2}\iiint\limits_{(V)}\rho G_{ik}\dot{u}^i\dot{u}^k\,dV. \tag{11.3}$$

The element of volume dV of the deformed body is connected with the element of volume dV_0 of the body before deformation by the relation resulting from eq.(6.1):

$$dV=(1+2g^{ik}D_{ik}+\ldots)\,dV_0. \tag{11.4}$$

The limits of integration in eqs.(11.2a) - (11.3) likewise depend on the components of the strain tensor.

Section 12. Work and Reciprocity Theorem in Nonlinear Elasticity Theory

The theorem of reciprocal work in the linear theory of elasticity is a consequence of the identity of two invariants associated with two states of the body:

$$A_0=\sigma^{ik}\varepsilon'_{ik}=\sigma'^{ik}\varepsilon_{ik}. \tag{12.1}$$

The identity (12.1) results from Hooke's law (4.1a) on replacing the tensor D_{rs} by the tensor of small deformations ϵ_{rs}.

We have indicated elsewhere (Bibl.23c) general considerations permitting /79 various generalizations of the work and reciprocity theorem to be found. In order to find a generalization of the reciprocal theorem to the nonlinear elasticity theory, let us make use of eq.(4.7), represented in the following form:

$$\sigma_*^{ik}=\sigma^{ik}-T^{ik}=C_1^{ik,\,rs}\varepsilon_{rs}, \tag{12.2a}$$

where

$$T^{ik}=\frac{1}{2}C_1^{ik,\,rs}\nabla_r u^j\nabla_s u_j+C_2^{ik,\,pq,\,rs}D_{pq}D_{rs}. \tag{12.2b}$$

We shall call the stresses σ_*^{ik} reduced stresses. Consider the invariant

$$A_{12}=\sigma_*'^{ik}\varepsilon'_{ik}. \tag{12.3}$$

In the case of small deformations, the invariant A_{12} passes over into the invariant A_0. From eqs.(12.2a) follows the identity:

$$A_{12}=A_{21}$$

or

$$\sigma_*^{\prime ik}\varepsilon''_{ik}=\sigma_*^{\prime\prime ik}\varepsilon'_{ik}. \tag{12.4}$$

It will be clear from eqs.(11.2a) that eq.(12.4) expresses the property of reciprocity of the work done by the reduced stresses of one state of an elastic body on the strains of the other state. This work may be referred, for example, to unit volume of the undeformed body.

We emphasize that the reference of the scalar A_{12} to unit volume of the undeformed body is arbitrary. In exactly the same way, one might use the unit volume of the body in the first or second state. To this arbitrary choice correspond three possible integral statements of the generalized reciprocal theorem. Multiplying eq.(12.4) by the volume element dV_0 of the undeformed body and integrating over the volume V_0, we find

$$\iiint\limits_{(V_0)}\sigma_*^{\prime ik}\varepsilon''_{ik}\,dV_0=\iiint\limits_{(V_0)}\sigma_*^{\prime\prime ik}\varepsilon'_{ik}\,dV_0. \tag{12.5}$$

Consider the integral

$$I=\iiint\limits_{(V_0)}\sigma_*^{\prime ik}\varepsilon''_{ik}\,dV_0. \tag{a}$$

We have

$$I=\frac{1}{2}\iiint\limits_{(V_0)}\sigma_*^{\prime ik}(\nabla_i u''_k+\nabla_k u''_i)\,dV_0=\iiint\limits_{(V_0)}\sigma_*^{\prime ik}\nabla_i u''_k\,dV_0. \tag{b}$$

Further, /80

$$I=\iiint\limits_{(V_0)}\sigma_*^{\prime ik}\nabla_i u''_k\,dV_0=\iiint\limits_{(V_0)}\nabla_i(\sigma_*^{\prime ik}u''_k)\,dV_0-\iiint\limits_{(V_0)}u''_k\nabla_i\sigma_*^{\prime ik}\,dV_0, \tag{12.6}$$

and, applying the Ostrogradskiy-Gauss formula, we find

$$\iiint\limits_{(V_0)}\nabla_i(\sigma_*^{\prime ik}u''_k)\,dV_0=\iint\limits_{(S_0)}\vec{F}'_n\cdot\vec{u}''\,dS_0; \tag{12.7}$$

with

$$\vec{F}_n = \overset{2}{\vec{\sigma}}_{\bullet} \cdot \vec{n}_0, \tag{12.8}$$

where $\vec{n}_0$ is the unit vector of the external normal to the surface S_0 of the undeformed body.

Using eqs.(5.2a) and the relation (6.15a), we get

$$\nabla_i \sigma_{\bullet}^{ik} = \rho \frac{\partial^2 u^k}{\partial t^2} - \rho F^k - P_{ij}{}^{l} \sigma_{\bullet}^{jk} - P_{ij}{}^{k} \sigma_{\bullet}^{ij} - \nabla_i T^{ik} = -\Phi^k. \tag{12.9}$$

The vector $\vec{\Phi}$ may be regarded as the force related to unit volume of the undeformed body. In this case, however, it must not be forgotten that all the kinetic quantities in eq.(12.9) are connected with an element of volume of the deformed body.

On the basis of eqs.(12.6) - (12.9), eq.(12.5) takes the following form:

$$\iint_{(S_0)} \vec{F}_n' \cdot \vec{u}'' dS_0 + \iiint_{(V_0)} \vec{\Phi}' \cdot \vec{u}'' dV_0 = \iint_{(S_0)} \vec{F}_n'' \cdot u' dS_0 + \iiint_{(V_0)} \vec{\Phi}'' \cdot \vec{u}' dV_0. \tag{12.10}$$

Equation (12.10) may be regarded as the formal generalization of the theorem of work and reciprocity of the linear theory of elasticity to the problem of the mechanics of anisotropic elastic bodies with physical and geometrical nonlinearity. In fact eq.(12.10), at small deformations and in the absence of inertial forces, yields the classical theorem of work and reciprocity.

We note in conclusion that the use of other methods for selecting the initial invariant A_{12}, would yield other integral equations which would likewise generalize, in the above sense, the reciprocal theorem of the linear theory of elasticity. All these generalizations do not literally correspond to the classical theorem since they contain quantities which are only by convention termed by us "surface" and "body" forces. The question of the possibility of proving the reciprocal theorem, free of these arbitrary elements, still remains open.

Section 13. Elastic Medium with Initial Stresses /81

In many cases it is necessary to investigate the deformation of elements of an elastic body in an already established stressed state.

The book by A.Love contains examples of cases in which the initial stresses

cannot be neglected. One of these examples is taken from the theory of shells*.

The modern practice of designing reinforced concrete and steel structures with prestressed elements likewise furnishes numerous examples from the field of mechanics investigated in the present study and demonstrates the necessity of formulating a general theory permitting a sufficiently rigorous mathematical analysis of these problems and similar ones.

The question of the relations between the components of the tensor of additional stresses and the tensor of additional deformations was posed long ago.

Love indicates that, to establish the relations between the additional stresses and strains, we must turn to a more general theory than that developed in the classical mechanics of elastic bodies, or to practical experiments. The outmoded theory of Cauchy and Green is obviously insufficiently substantiated*.

We present below the relations between the additional stresses and the additional strains resulting from relations (4.1a) containing geometrically nonlinear terms.

We shall assume that there exists an initial undeformed state of the body. The body is then deformed and the initial stresses σ_0^{ik} and displacements u_{0i} appear and are related by eqs.(4.1a):

$$\sigma_0^{ik} = \frac{1}{2} C^{ik,rs} (\nabla_r u_{0s} + \nabla_s u_{0r} + \nabla_r u_0^j \nabla_s u_{0j}). \tag{13.1}$$

Equation (13.1) corresponds to the mechanical methods of establishing initial stresses. If the initial stresses are due to thermal effects, eqs.(13.1) must be supplemented by temperature-dependent terms.

Consider certain consequences resulting from eqs.(13.1). As a result of the additional strain, let new stresses and displacements arise, connected with their initial values by the relations

$$\sigma^{ik} = \sigma_0^{ik} + T^{ik}; \quad u_i = u_{0i} + v_i. \tag{13.2}$$

It is here assumed that the components v_i are small in absolute value, i.e., small in relation to a certain characteristic measurement of the body. In the shell theory, such a quantity is the thickness of the shell. /82

Substituting eqs.(13.2) into eqs.(4.1a) we obtain, after a number of transformations and discarding the terms that are nonlinear in v_i and the derivatives of v_i:

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* Cf. A.Love, Mathematical Theory of Elasticity (Russian Translation) ONTI, 1935, pp.120 - 122.

$$T^{ik} = (C^{ik,rs} + C^{ik,js}\nabla_j u_0^r)\nabla_s v_r. \tag{13.3}$$

Symmetrizing the left-hand side of this equation in the indices r and s, we find

$$T^{ik} = \frac{1}{2}(C^{ik,rs} + C^{ik,js}\nabla_j u_0^r + C^{ik,jr}\nabla_j u_0^s)(\nabla_s v_r + \nabla_r v_s). \tag{13.4}$$

Let

$$\varkappa^{ik,rs} = C^{ik,rs} + C^{ik,js}\nabla_j u_0^r + C^{ik,jr}\nabla_j u_0^s, \tag{13.5}$$

$$d_{rs} = \frac{1}{2}(\nabla_s v_r + \nabla_r v_s), \tag{13.6}$$

where d_{rs} is the tensor of small additional deformations. Then, eqs.(13.4) take on the form of a generalized Hooke's law:

$$T^{ik} = \varkappa^{ik,rs} d_{rs}. \tag{13.7}$$

The quantities $\varkappa^{ik,rs}$ may be considered as being components of the elastic tensor in a body with initial stresses.

In eq.(13.5) we replace the tensor components $\nabla_j u_0^r$ by their expressions in terms of the tensor components ϵ_{ik} and Ω_{ik}, resulting from eqs.(2.6) - (2.7). We then find

$$\varkappa^{ik,rs} = C^{ik,rs} + C^{ik,js}\epsilon_{j.}^{\cdot r} + C^{ik,jr}\epsilon_{j.}^{\cdot s} + C^{ik,js}\Omega_{j.}^{\cdot r} + C^{ik,jr}\Omega_{j.}^{\cdot s}. \tag{13.8}$$

It is clear from eqs.(13.5) and (13.8) that, under large displacements of $\vec{u}_0$ or of components of the strain tensor $\vec{\epsilon}$ and of the tensor Ω connected with rotations of elements of the body, a prestressed body on subsequent deformations must be regarded as an inhomogeneous body with varied anisotropy. In particular, an isotropic body is converted into an anisotropic body. These facts are also known from geometrical optics, but in solving the problems of the mechanics of elastic bodies they are of substantial importance only in displacements of $\vec{u}_0$ high in modulus, for great absolute values of the components of the tensors $\vec{\epsilon}$ and Ω. All above statements also apply to physically nonlinear elastic bodies, for which the relations (4.7) are valid.

CHAPTER III /83

REDUCTION OF THE THREE-DIMENSIONAL PROBLEMS OF THE MECHANICS OF ELASTIC BODIES TO THE TWO-DIMENSIONAL PROBLEMS OF THE THEORY OF SHELLS

Section 1. General Characterization of the Problem

The solution of the three-dimensional problems of elasticity theory involves considerable mathematical difficulties. For this reason, long ago, during the very development of the methods for solving the problems of elasticity theory, two groups of problems were distinguished, permitting the substitution of systems of elastodynamic equations by systems of approximate equations containing a smaller number of independent variables than the original equations. This decrease in the number of independent variables is equivalent to decreasing the number of dimensions of space, since the independent variables in the equations of elasticity theory are the space coordinates and time.

The two mentioned groups of problems are the problems of the motion of elements of thin elastic rods and those of the dynamically stressed and strained states of shells.

In the former case, the relations of two spatial measurements of the body to the third dimension are negligible, so that the three-dimensional problem of the elasticity theory can be reduced to a one-dimensional problem.

In the shell theory, it is assumed that the ratio of one of the dimensions of the body - the thickness of the shell - to the other dimensions is small. Then, as we shall show, the three-dimensional problem of the theory of elasticity can be approximately reduced to a two-dimensional problem.

The ratio of the thickness of a shell to one of the characteristic parameters determining the dimensions of the shell is limited by various conditions in conventional studies of the subject matter. These conditions depend primarily on the accuracy of the approximate representation of the three-dimensional dynamic boundary problems of the two-dimensional elasticity theory. It is obvious that the boundary conditions of the problem are of great importance here. It is therefore impossible to set up any general absolute criterion which the thickness of a shell must satisfy, to ensure a predetermined accuracy in the /84 solution of equations approximately describing its state.

We give below a brief survey of the present methods of classifying shells according to their thickness. This classification also involves concepts on the limits of applicability of various methods of analytical description of the dynamically stressed and strained state of a shell.

We will make some preliminary remarks* on the general problem of reducing

* We recall that there are two-dimensional problems in the elasticity theories that are not connected with shell theory.

the three-dimensional problem of the elasticity theory to a two-dimensional problem, under the assumption that the shell is an elastic body. Let us select on the base surface of an undeformed shell an arbitrary coordinate system x^i (i= = 1,2). The coordinate vector $\vec{e}_3$ in the undeformed shell will be taken, according to (I, Sect.3) as equal to the unit vector of the normal $\vec{n}$ to the base area. The vectors $\vec{e}_i$ (l = 1, 2, 3) form the local coordinate base. We agree that the mutual orientation of these vectors corresponds to a right-hand coordinate system.

The general program of reduction of the three-dimensional problem of the theory of elasticity to a two-dimensional problem consists in constructing analytic expressions for the quantities characterizing the stressed and strained state of the shell in terms of new quantities determined in the coordinates x^i (i = 1, 2) of its base area, and in setting up the equations that these quantities must satisfy in the region of variation of the variables x^i and on their boundaries.

The equations of shell theory might be said to describe the stressed and strained state of the base area. Obviously, these equations must not contain derivatives with respect to the coordinate x^3.

As we shall show later, the "reduction problem" has no unique solution*. But the solution, on the other hand, cannot be completely arbitrary. It is restricted by the requirements of optimum approximate representation of the equations of elasticity theory by the equations of shell theory. It is well known from the theory of approximation functions that the concept "optimum approximation" is not entirely definite. For example, there exist optimum approximations at a given point of a manifold to which an approximation function is assigned, optimum approximations in the mean in a certain region of variation of its arguments, etc.. To different methods of approximation functions there correspond different methods of approximate reduction of the three-dimensional /85
problems of elasticity theory to two-dimensional problems.

The solution of the reduction problem depends largely on the choice of the approximation method for the components τ^{i3} of the stress tensor or D_{i3} of the strain tensor, considered as functions of the coordinate x^3.

Section 2. Remarks on the Methods of Reduction given by Poisson, Cauchy, Kirchhoff, and Love

General methods for the reduction of a three-dimensional static problem of the elasticity theory to a two-dimensional problem were developed by Poisson and Cauchy, in considering the equilibrium of a plate. These great mathematicians applied the expansion of the stress tensor components in ascending powers of the coordinate z, measured along a normal to the undeformed middle plane of the plate. Using the equations of equilibrium of an element of a continuous medium, and assuming that the boundary planes of the plate were free of loads, Cauchy and Poisson obtained a fundamental system of equations and boundary conditions of the boundary problem for the equilibrium of a plate. The Cauchy-

* Here and hereafter, for brevity, we will use the term "reduction problem".

Poisson method was subjected to a critical analysis by Saint-Venant, Kirchhoff, and several other investigators. Saint-Venant noted that it was not fully justified to expand stress tensor components not known in advance into series in powers of even the relatively small coordinate z. These series, in his opinion, might converge in a sufficiently small neighborhood of an interior point of the plate, but their convergence over the entire range of variation of the coordinate z might not take place. He referred in this connection to the inaccurate results obtained by the Cauchy and Poisson methods in the theory of the torsion of prisms. For this reason, he preferred different methods of setting up the fundamental system of equations of the theory of plates, including the Kirchhoff method, based on well-known simplifying hypotheses*.

We shall not dwell here on a discussion of the boundary conditions in the theory of plates, since we will revert to this subject later. Saint-Venant's objections to the Cauchy and Poisson methods, to a considerable degree, reflected the state of the theory of elasticity in the third quarter of the last Century. It is known that, at that time, only the foundations of the general solution methods for boundary problems of elasticity theories were being prepared, permitting conclusions from the analytical properties of these solutions.

In the last quarter of the 19th Century, the work done by Somigliano, Volterra, and Lauricella led to the conclusion that, in the absence of body forces, the solution of the problems of the statics of an elastic body are analytic /86
functions of the coordinates of internal points of the body, i.e., these solutions can be expanded in series in positive powers of the coordinates**. But the convergence of these expansions on the surface of the body requires a separate investigation for specific problems. Moreover, the conclusions on the analytic properties of a solution of static problems can be extended to the dynamics of elastic bodies only after additional analysis.

We shall now present facts confirming the significance of the method under discussion. We recall that the method of the preliminary introduction of expansions in series, used by Poisson and Cauchy in the theory of plates, is encountered even today in various fields of the elasticity theory, for example in the static plane problem. Here, on the basis of the analytic properties of solutions of the plane problem, expansions in power series are introduced, and then their convergence over the entire region of determination of the required functions, including their boundary, is confirmed in special cases***. The method of expansion of the required functions in series in positive powers of the coordinate x^3, as could have been predicted, was found to be an effective means of constructing the theory of thick plates (Bibl.9b, 25b).

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* Saint-Venant's criticism of the work of Poisson and Cauchy is given by Clebsch in the book "Theory of Elasticity of Solid Bodies", Paris, 1883, pp.722 - 725.

** Cf., for example, E.Trefftz, Mathematical Theory of Elasticity, ONTI, 1934, p.135.

*** Cf. N.I.Muskhelishvili, Some Fundamental Problems of the Mathematical Theory of Elasticity, USSR Academy of Sciences, 1949.

All the above permits us to assert that Saint-Venant's criticism of the Poisson and Cauchy methods is not sufficient reason for abstaining from a further development of these methods with respect to the theory of shells. Proper caution must, however, be exerted in the special cases noted below.

The Kirchhoff theory was subsequently extended by A.Love to include the theory of shells. The Kirchhoff-Love theory is based on the well-known postulate that the normal to the undeformed middle surface of a shell remains normal to it even after deformation. This hypothesis is supplemented by one of two hypotheses about the variation in its length.

According to the first of these hypotheses, a segment of a normal to the middle surface enclosed within a shell does not vary its length under deformation of the shell. In this case, the Kirchhoff-Love hypothesis is appropriately called the "hypothesis of straight constant normals".

According to the second version it is assumed that the normal stresses σ_{x3} are small by comparison with $\sigma_{x_i x_k}$ $(i,k = 1,2)$ and can be neglected*. We /87
note that the first hypothesis is not equivalent to the second, a fact which is not made sufficiently clear in certain well-known monographs.

The "hypothesis of an invariant normal" naturally leads to a replacement of the vectors of stresses acting at the boundary of a shell element with generatrices normal to the middle surface by the statically equivalent system of forces applied to the contour of an element of the middle surface. We emphasize that this substitution of the actual system of stresses by a system of forces and moments statically equivalent to it is internally in harmony with the "hypothesis of straight and invariant normals"**.

In the remaining cases this agreement does not appear. The Kirchhoff-Love hypotheses introduce an unremovable error into the equations of shell theory, and this error must be taken into account in evaluating the possibility of various simplifications of the equations of shell theory.

Section 3. Preliminary Classification of Shells Connected with the Kirchhoff-Love Hypotheses. Linear and Nonlinear Problems

The work by V.Novozhilov and R.Finkel'shteyn (Bibl.28) gives an estimate of the error introduced by the Kirchhoff-Love hypothesis into the equations of shell theory. It was shown that this error is of the order of $\max(2hk_i)$, where k_i is one of the principal curvatures of the shell. This estimate permits us to distinguish the class of shells for which the equations of classical shell theory based on the Kirchhoff-Love hypothesis are still sufficiently accurate, permitting, for example, a determination of their stress fields with a relative error not exceeding 5%. These shells are called thin. All other shells will

* Here $\sigma_{x_i x_k}$ are the "physical components" of the stress tensor. Cf.(I, Sect.5) and (I, Sect.7).

** Here and hereafter we assume the reader to be familiar with the theory of shells, for example to the extent given in (Bibl.11a). For this reason, we use certain terms without first giving their definition.

here be called thick shells or shells of medium thickness*. Let us define these ideas using the definitions given in several modern studies. We introduce the notation (Bibl.1):

$$\max(2hk_i) = \varepsilon; \quad \max\left(\frac{2h}{a}\right) \doteq \eta, \tag{3.1}$$

where a is one of the parameters determining the dimensions of the basic surface of the shell.

The classification of shells into thin and thick is related primarily to the quantity ϵ. Most often, because of the fact that the relative error of the solution of approximate equations considered in shell theory is restricted to 5%, a shell is called thin if the condition (Bibl.1, 11a) /88

$$\varepsilon \leqslant \frac{1}{20}. \tag{3.2}$$

is satisfied.

If the condition (3.2) is not satisfied, the shell is called thick. Of course, the condition (3.2) is somewhat arbitrary, since a rigorous estimate of the error of solutions of the equations of shell theory, constructed on the basis of simplifying assumptions with various boundary conditions, is very difficult.

The upper limit of values of η has been insufficiently studied. S.A.Ambartsumyan (Bibl.1) assumes in his book that $\eta \leq 0.1$, while A.S.Vol'mir in his book (Bibl.4) states that $\eta \leq 0.2$. The quantity η can obviously not be determined without a delimitation of the class of the boundary problems of shell theory.

K.Z.Galimov and Kh.M.Mushtary in their monograph (Bibl.10) have pointed out a different approach to the classification of shells, obviously based on physical considerations. A shell, according to these authors (Bibl.10), is called thin if it satisfies the condition

$$2hL^{-1} \leqslant \varepsilon_p. \tag{3.3}$$

Whereas, if it satisfies the condition

$$\varepsilon_p \leqslant 2hL^{-1} \leqslant \sqrt{\varepsilon_p} \tag{3.4}$$

* The method of evaluating the error introduced by the Kirchhoff-Love hypothesis, advanced by Novozhilov and Finkel'shteyn, has evoked critical remarks by V.M.Darevskiy (Bibl.22).

a shell is of medium thickness. Here L is a linear dimension, characteristic for a shell or a plate, for example one of the principal radii of curvature of the basic surface or its smallest diameter. The quantity ε_p is the relative elongation corresponding to the proportional limit of the material of the shell.

A number of investigators suggest that no preliminary restrictions be imposed on the thickness of the shell. We shall return later to this question, when we base our own classification of shells, as a function of their thickness, on the theory of propagation of dynamic wave processes in such shells.

It is well known that plates and shells are elastic bodies in which the displacements, deformations, and angles of rotation of the elements may be so great that an application of the linear equations of the classical theory of elasticity would lead to substantial errors. In such cases, nonlinear equations must be used.

In determining the boundaries of applicability of the linear theory, one usually starts out from the ratio of the displacements of the points of the basic surface of the shell to its thickness. This method of classification, however, is arbitrary, since here too the decisive influence is that of the preassigned error limits in the solution of the boundary conditions of the theory of shells. It may be considered, according to A.S.Vol'mir, that the linear /89
theory is applicable if $|\vec{u}| : 2h \leq 1/5$, and is entirely inapplicable if $|\vec{u}| : 2h \geq 5$ (Bibl.4) where $\vec{u}$ is the displacement vector of the basic surface of the shell.

K.Z.Galimov and Kh.M.Mushtary give a different approach to the criteria of applicability of the linear theory. They distinguish weak, medium, and strong flexures of the shell.

A weak flexure takes place if the following condition is satisfied in the shell:

$$\max|\vec{\omega}| \ll 1, \tag{3.5}$$

where $|\vec{\omega}|$ is the modulus of the vector of rotation of an arbitrary linear element under flexure of the shell. It is found here that, for some classes of boundary problems, the condition (3.5) is satisfied if

$$\max|\vec{\omega}| \approx 2h; \tag{3.6a}$$

for other types of boundary problems, the condition (3.5) leads to the inequation

$$\max|\vec{u}| \leqslant 2h\sqrt{\varepsilon_p}. \tag{3.6b}$$

For a moderate flexure, the displacement of points of the basic surface, by modulus, equals or exceeds 2h but is considerably smaller than the other linear

dimensions of the shell. Here,

$$\max|\vec{\omega}|^2 \ll 1. \tag{3.7}$$

The relation (3.7) gives us the right to neglect quantities of the order of $|\vec{\omega}|^2$ by comparison with unity.

In a strong flexure, the displacements, directed along the normal to the basic surface*, are great relative to 2h and are of the order L. In this case, likewise, the quantities $|\vec{\omega}|^2$ will be great.

The problems related to a weak flexure are described by linear systems of equations, while those of moderate and strong flexure belong to the nonlinear theory of shells. If we confine ourselves to the study of elastic deformations, then we must impose on the problems of moderate and strong flexure additional restrictions that ensure the absence of zones of plasticity. Finally, these restrictions must be connected with methods for the reduction of a three-dimensional problem of the theory of elasticity to a two-dimensional problem and with specific boundary conditions. It is difficult to indicate these restrictions in the general form for extensive classes of problems.

Section 4. Application of Tensor Series. Reduction of the Three-Dimensional Problem to the Determination of an Infinite Sequence of Functions of a Point of the Base Area of the Shell /90

Let us extend the methods applied by Cauchy and Poisson in the statics of plates to the elastodynamic problems of shell theory. To bring out the fundamental ideas of the method, let us first consider the linear equations of shell theory, holding to the exposition adopted by us in other work (Bibl.23a, b). We will use expansions in tensor series, which are generalized Poisson and Cauchy series. We emphasize that, instead of tensor series, expansions in ordinary power series can also be used (Bibl.23a, b and 26). Expansions in tensor series, in our opinion, have the following advantages:

a) Such expansions lead to equations valid in an arbitrary curvilinear system of coordinates, which is particularly convenient in solving nonlinear problems;

b) Each term in the approximation formulas is a quantity with a definite geometrical meaning. The latter permits a clearer and more pictorial presentation of the meaning of various simplifications of the equations than the use of conventional expansions in powers of the coordinate x^3.

Hereafter, we will consider several versions of the generalized expansions in power series. Let us make use of eq.(I, 12.3). Assume, neglecting the strains in the shell, that

$$(\Delta\vec{r})^1 \overset{*}{=} (\Delta\vec{r})^2 \overset{*}{=} 0; \quad (\Delta\vec{r})^3 \overset{*}{=} z. \tag{4.1}$$

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* These displacements are ordinarily called flexural.

The sign $\overset{*}{=}$, here and hereafter, denotes an equality that is true only in one definite coordinate system (a non-invariant equality). The expressions of the vector components $\Delta\vec{r}$ in the deformed shell will be given below when we discuss the nonlinear theory*.

Following the general program of solution of the reduction problem (Section 1), let us consider the expansions of the strain tensor components into tensor series, remembering eq.(4.1):

$$D_{ik}^{(z)} \overset{*}{=} D_{ik} + z\nabla_3 D_{ik} + \frac{1}{2} z^2 \nabla_3\nabla_3 D_{ik} + \dots \quad . \tag{4.2}$$

where $D_{ik}^{(z)}$ are the components of the strain tensor at the point with the coordinate $x^3 = z$, while D_{ik} are the strain tensor components on the basic surface of the shell.

We note that, according to eq.(I, 12.3), an expression of the form

$$z^m \underbrace{\nabla_3\nabla_3 \dots \nabla_3}_{m \text{ times}} D_{ik}$$

is to be regarded as a component of a covariant tensor of second rank, and an /91 expression of the form

$$z^m \underbrace{\nabla_3 \dots \nabla_3}_{m \text{ times}} u_k$$

as a component of a covariant vector.

Let us assume that the tensor of curvature vanishes in the Lagrangian system of coordinates x^i associated with the undeformed or deformed shell. Then, according to eq.(I, 10.9), we have the right to change, in the multiple covariant derivatives, the sequence of operations of differentiation. To simplify the calculations, let us assume that the system of coordinate lines x^i (i = 1,2) on the basic surface coincides with its lines of curvature. Using eqs.(I,3.6a) - (3.6b), (9.5) and (9.8), we find the nonzero Christoffel symbols of index 3.

We have

$$\Gamma_{i3}^{i} = -\frac{k_i}{1-k_i z}; \quad \Gamma_{ii}^{3} = (g_{ii})_0 k_i (1-k_i z) \tag{4.3}$$

* See also (Bibl.23b, Part II, Sect.4).

or, for z = 0,

$$\Gamma^i_{i3} = -k_i; \quad \Gamma^3_{ii} = (g_{ii})_0 k_i \qquad (i = 1, 2). \tag{4.4}$$

Hereafter, to shorten the formulas, we shall denote the components of the metric tensor on the basic surface by g_{ik} and at an arbitrary point by $g^{(z)}_{ik}$.

One more remark: The functions z^m are components of the tensor $(\Delta\vec{r})^p(\Delta\vec{r})^q\ldots$. From eqs.(4.1) and (4.3) it follows that

$$\nabla_i(\Delta\vec{r})^3 \overset{*}{=} 0 \qquad (i = 1, 2).$$

But

$$\nabla_i(\Delta\vec{r})^j \neq 0 \quad \text{at} \quad j = 1, 2,$$

in spite of the first two relations of eqs.(4.1). For this reason, in covariant differentiation, the quantities z cannot be regarded as constants.

Bearing in mind all that has been said, we find, in expanded form, the expansions of the tensor components of small deformation in tensor series in powers of z. We have

$$\begin{aligned}\varepsilon^{(z)}_{11} = \partial_1 u_1 - \Gamma^1_{11}u_1 - \Gamma^2_{11}u_2 - g_{11}k_1u_3 + z\,[\partial_1\nabla_3 u_1 + k_1\nabla_1 u_1 - \\ - \Gamma^1_{11}\nabla_3 u_1 - \Gamma^2_{11}\nabla_3 u_2 - g_{11}k_1\nabla_3 u_3] + \ldots;\end{aligned} \tag{4.5a}$$

$$\begin{aligned}2\varepsilon^{(z)}_{12} = \partial_1 u_2 + \partial_2 u_1 - 2\Gamma^1_{12}u_1 - 2\Gamma^2_{12}u_2 + z\,[\partial_1\nabla_3 u_2 + \partial_2\nabla_3 u_1 + \\ + k_1\nabla_1 u_2 + k_2\nabla_2 u_1 - 2\Gamma^1_{12}\nabla_3 u_1 - 2\Gamma^2_{12}\nabla_3 u_2] + \ldots;\end{aligned} \tag{4.5b}$$

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$$\begin{aligned}\varepsilon^{(z)}_{22} = \partial_2 u_2 - \Gamma^1_{22}u_1 - \Gamma^2_{22}u_2 - g_{22}k_2u_3 + z\,[\partial_2\nabla_3 u_2 + \\ + k_2\nabla_2 u_2 - \Gamma^1_{22}\nabla_3 u_1 - \Gamma^2_{22}\nabla_3 u_2 - g_{22}k_2\nabla_3 u_3] + \ldots;\end{aligned} \tag{4.5c}$$

$$2\varepsilon^{(z)}_{13} = \nabla_1 u_3 + \nabla_3 u_1 + z\,[\partial_1\nabla_3 u_3 + k_1(\nabla_1 u_3 + \nabla_3 u_1) + \nabla_3\nabla_3 u_1] + \ldots; \tag{4.5d}$$

$$2\varepsilon^{(z)}_{23} = \nabla_2 u_3 + \nabla_3 u_2 + z\,[\partial_3\nabla_3 u_3 + k_2(\nabla_2 u_3 + \nabla_3 u_2) + \nabla_3\nabla_3 u_2] + \ldots; \tag{4.5e}$$

$$\varepsilon^{(z)}_{33} = \nabla_3 u_3 + z_3\nabla_3\nabla_3 u_3 + \ldots\,. \tag{4.5f}$$

where

$$\nabla_3 u_j = \partial_3 u_j + k_j u_j \qquad (j = 1, 2); \quad \nabla_3 u_3 = \partial_3 u_3. \tag{4.6}$$

Substituting eqs.(4.5a) - (4.5f) into eqs.(II, 4.1a) or into eqs.(II,4.5a) - (II, 4.5b) and rejecting the nonlinear terms, we obtain expansions of the stress tensor components in powers of z.

In spite of the fact that all the coefficients of the z^m in the resultant expansion are functions of the coordinates x^i, these expansions do not yet solve the reduction problem, since they contain an infinite set of derivatives of the form $\partial_3 \dots \partial_3 u_j$ or of covariant derivatives $\nabla_3 \dots \nabla_3 u_j$. These derivatives may be regarded as new unknown functions subject to determination. The reduction problem will be solved after eliminating the derivatives $\nabla_3 \dots \nabla_3 u_j$ from the system of equations of shell dynamics.

Section 5. Reduction of the Three-Dimensional Problem to the Determination of Six Functions of a Point of the Base Area of the Shell

We shall show that the reduction problem has a definite analytic meaning, i.e., that it can be formulated as a problem of mathematical physics. Indeed, the Lamé equations (II, 5.5b) permit us, as was first shown elsewhere (Bibl.23a and b), to express on the basic surface the derivatives $\nabla_3\nabla_3 \dots u_j$, beginning with the derivative of second order, in terms of the derivatives $\nabla_3 u_j$, of functions of u_j, and of the "tangential" derivatives $\nabla_i\nabla_k \dots \nabla_r u_j$, i.e., derivatives with respect to the coordinates x^1 and x^2 of the basic surface. From eq.(II, 5.5b) we find:

$$\nabla_3\nabla_3 u_j \overset{*}{=} -\frac{\lambda+\mu}{\mu} g^{rr}\nabla_i\nabla_r u_r - g^{ss}\nabla_s\nabla_s u_i - \frac{\rho}{\mu}F_i + \frac{\rho}{\mu}\frac{\partial^2 u_i}{\partial t^2}, \tag{5.1a}$$

$$\nabla_3\nabla_3 u_3 \overset{*}{=} -\frac{\lambda+\mu}{\lambda+2\mu} g^{ss}\nabla_s\nabla_3 u_s - \frac{\mu}{\lambda+2\mu} g^{ss}\nabla_s\nabla_s u_3 - \frac{\rho}{\lambda+2\mu}F_3 + \frac{\rho}{\lambda+2\mu}\frac{\partial^2 u_3}{\partial t^2} \qquad (i,\ s=1,\ 2;\ r=1,\ 2,\ 3). \tag{5.1b}$$

Further, differentiating eqs.(5.1a) - (5.1b) we obtain /93

$$\nabla_3\nabla_3\nabla_3 u_i \overset{*}{=} -\frac{\lambda+\mu}{\mu}(g^{ss}\nabla_i\nabla_s\nabla_3 u_s + \nabla_i\nabla_3\nabla_3 u_3) - g^{ss}\nabla_s\nabla_s\nabla_3 u_i - \frac{\rho}{\mu}\nabla_3 F_i + \frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}\nabla_3 u_i; \tag{5.2a}$$

$$\nabla_3\nabla_3\nabla_3 u_3 \overset{*}{=} -\frac{\lambda+\mu}{\lambda+2\mu} g^{ss}\nabla_s\nabla_3\nabla_3 u_s - \frac{\mu}{\lambda+2\mu} g^{ss}\nabla_s\nabla_s\nabla_3 u_3 - \frac{\rho}{\lambda+2\mu}\nabla_3 F_3 + \frac{\rho}{\lambda+2\mu}\frac{\partial^2}{\partial t^2}\nabla_3 u_3 \qquad (i,\ s=1,\ 2). \tag{5.2b}$$

Substituting into the resultant expression the values of the second derivatives $\nabla_3\nabla_3 u_i$ from eqs.(5.1a) - (5.1b), we find the final expressions for the derivatives:

$$\nabla_3\nabla_3\nabla_3 u_i \overset{*}{=} g^{ss}\left[\frac{\lambda+\mu}{\lambda+2\mu}(\nabla_i\nabla_s\nabla_s u_3 - \nabla_i\nabla_s\nabla_3 u_s) - \nabla_s\nabla_s\nabla_3 u_i\right] + \\ + \frac{(\lambda+\mu)\rho}{\mu(\lambda+2\mu)}\nabla_i F_3 - \frac{\rho}{\mu}\nabla_3 F_i - \frac{(\lambda+\mu)\rho}{\mu(\lambda+2\mu)}\times \\ \times\frac{\partial^2}{\partial t^2}\nabla_i u_3 + \frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}\nabla_3 u_i; \tag{5.3a}$$

$$\nabla_3\nabla_3\nabla_3 u_3 \overset{*}{=} \frac{\lambda+\mu}{\mu} g^{rr} g^{ss}\nabla_r\nabla_r\nabla_s u_s + \frac{\lambda}{\mu} g^{rr}\nabla_r\nabla_r\nabla_3 u_3 + \frac{(\lambda+\mu)\rho}{\mu(\lambda+2\mu)}\times \\ \times g^{rr}\nabla_r F_r - \frac{\rho}{\lambda+2\mu}\nabla_3 F_3 - \frac{(\lambda+\mu)\rho}{\mu(\lambda+2\mu)}\times \\ \times g^{rr}\frac{\partial^2}{\partial t^2}\nabla_r u_r + \frac{\rho}{\lambda+2\mu}\frac{\partial^2}{\partial t^2}\nabla_3 u_3 \\ (r,\ s=1,\ 2). \tag{5.3b}$$

Differentiating eqs.(5.3a) - (5.3b) and repeating the process of eliminating the derivatives $\nabla_3 \dots \nabla_3 u_j$, beginning with the derivatives of second order, we can find expressions for the derivatives $\nabla_3 \dots \nabla_3 u_j$ of arbitrary multiplicity in terms of derivatives containing the operator ∇_3 in an order not higher than the first. By means of the resultant expressions for the derivatives of $\nabla_3 \dots \nabla_3 u_j$, all quantities characterizing the stressed and strained state of the shell will be determined by expansions of the form of eqs.(4.5a) - (4.5f), as functions of the coordinates of the basic surface x^i (i = 1, 2) and explicit functions of the coordinate $x^3 = z$, if we know the six functions x^i : x_j and $\nabla_3 u_j$. We shall consider below certain methods of determining these functions, which will be called fundamental. We draw the reader's attention to the increase in the order of the time derivatives entering into the equations, if terms containing z^4 are introduced into the expansions. /94

Section 6. Application of the Symbolic Method

The reader has probably noticed that the determination of the derivatives $\nabla_3 \dots \nabla_3 u_j$ in the preceding Section, beginning with derivatives of the second order, in terms of quantities determined in the internal system of coordinates of the basic surface and the derivatives of $\nabla_3 u_j$ is essentially an algebraic operation. This operation is simplified in connection with the Ricci theorem and the vanishing of the curvature tensor. We recall that the Ricci theorem (I, Sect.9) permits us to operate with components of the metric tensor as with constant quantities in covariant differentiation, while the vanishing of the curvature tensor permits us to vary the sequence of differentiation in multiple derivatives. These properties of the operations employed by us permit the use,

in determining the derivatives $\nabla_3 \dots \nabla_3 u_j$, of the symbolic methods developed by A.I.Lur'ye*. Let us introduce the notation:

$$u_j^{(n)} = \underbrace{\nabla_3 \dots \nabla_3}_{n \text{ times}} u_j; \quad u_j^{(0)} = u_j. \tag{6.1a}$$

$$F_j^{(n)} = \underbrace{\nabla_3 \dots \nabla_3}_{n \text{ times}} F_j; \quad F_j^{(0)} = F_j. \tag{6.1b}$$

The Lamé equations in the form of eqs.(5.1a) - (5.1b) lead to the following relations:

$$u_i^{(n)} \stackrel{*}{=} -\frac{\lambda+\mu}{\mu} g^{ss} \nabla_i \nabla_s u_s^{(n-2)} - g^{ss} \nabla_s \nabla_s u_i^{(n-2)} - \frac{\lambda+\mu}{\mu} \nabla_i u_3^{(n-1)} + \frac{\rho}{\mu} \frac{\partial^2 u_i^{(n-2)}}{\partial t^2} - \frac{\rho}{\mu} F_i^{(n-2)}; \tag{6.2a}$$

$$u_3^{(n)} \stackrel{*}{=} -\frac{\lambda+\mu}{\lambda+2\mu} g^{ss} \nabla_s u_s^{(n-1)} - \frac{\mu}{\lambda+2\mu} g^{ss} \nabla_s \nabla_s u_3^{(n-2)} + \frac{\rho}{\lambda+2\mu} \times \frac{\partial^2 u_3^{(n-2)}}{\partial t^2} - \frac{\rho}{\lambda+2\mu} F_3^{(n-2)}$$

$$(i,\ s=1,\ 2; \quad n=2,\ 3, \dots). \tag{6.2b}$$

Equations (6.2a) - (6.2b), in a more easily visualized form than relations (5.1a) - (5.1b), show the recurrence of the relations between the successive covariant derivatives with respect to x^3 that result from the Lamé equations. /95

We shall now introduce abbreviated symbols for the differential operators. Let

$$L_i = -\frac{\lambda+\mu}{\mu} \nabla_i; \quad L_{i\cdot}^{\cdot s} = -\frac{\lambda+\mu}{\mu} g^{ss} \nabla_i \nabla_s;$$

$$M = -g^{ss} \nabla_s \nabla_s + \frac{\rho}{\mu} \frac{\partial^2}{\partial t^2}; \tag{6.3a}$$

* From the works of A.I.Lur'ye we here cite the monograph (Bibl.9b) in which this method was applied to the theory of equilibrium of an elastic layer. The Lur'ye method was also applied by I.T.Selezov in his dissertation "Study of the Propagation of Elastic Waves in Plates and Shells" (Institute of Mechanics Ukr SSR Academy of Sciences, 1961) in setting up the generalized equations for the transverse vibrations of plates, by the method developed in our own work (Bibl.23a, b).

$$M_3 = -\frac{\mu}{\lambda+2\mu} g^{ss}\nabla_s\nabla_s + \frac{\rho}{\lambda+2\mu}\frac{\partial^2}{\partial t^2} = \frac{\mu}{\lambda+2\mu} M;$$

$$N_{3.}^{\cdot s} = -\frac{\lambda+\mu}{\lambda+2\mu} g^{ss}\nabla_s. \tag{6.3b}$$

The operator M is known to us from the theory of propagation of waves.

Equations (6.2a) - (6.2b) take the following form:

$$u_i^{(n)} = L_i u_3^{(n-1)} + L_{i.}^{\cdot s} u_s^{(n-2)} + M u_i^{(n-2)} - \frac{\rho}{\mu} F_i^{(n-2)}, \tag{6.4a}$$

$$u_3^{(n)} = N_{3.}^{\cdot s} u_s^{(n-1)} + \frac{\mu}{\lambda+2\mu} M u_3^{(n-2)} - \frac{\rho}{\lambda+2\mu} F_3^{(n-2)}$$
$$(i,\ s=1,\ 2;\quad n=2,\ 3,\ldots). \tag{6.4b}$$

The system of eqs.(6.4a) - (6.4b) may be regarded as a system of algebraic equations permitting us to express successively all the functions of $u_i^{(n)}$ and $u_3^{(n)}$ in terms of $u_j^{(0)}$ and $u_j^{(1)}$ (j = 1, 2, 3). We shall not here perform this operation of successive elimination. The initial step of this operation had been pointed out in the preceding Section. We note, in conclusion, the tensor properties of the quantities introduced by us, and of eqs.(6.4a) - (6.4b).

If we consider point transformations of coordinates on the basic surface of a shell, then with respect to these transformations the quantities $u_i^{(n)}$ (i = = 1, 2) are vector components while the quantities $u_3^{(n)}$ are scalars. The proof of this assertion is obvious. The operators introduced by us can also be regarded as symbolic tensors of various ranks and structures on a set of coordinates x^i (i = 1, 2).

Section 7. Expressions for the "Normal" Part of the Stress Tensor. The Equations Determining the Fundamental Functions

We shall call the set of components σ^{i3} the normal part of the stress tensor. The other components of the stress tensor form its tangential part. It is easy to convince ourselves that the components σ^{i3} (i = 1, 2) are /96 vector components on the set of internal coordinates of the points of the basic surface, and that the component σ^{33} is a scalar on this set.

Making use of the expansions (4.5d) - (4.5f) extended to include terms in z^3, of the notations of eqs.(6.1a) and (6.1b), and of Hooke's law (II, 4.5b), we find:

$$\sigma_{i3} = \mu\left\{\nabla_i u_3 + u_i^{(1)} + z\,[\partial_i u_3^{(1)} + k_i(\nabla_i u_3 + u_i^{(1)}) + u_i^{(2)}] + \right.$$
$$\left. + \frac{1}{2} z^2 [\nabla_i u_3^{(2)} + u_i^{(3)}] + \ldots\right\}, \tag{7.1a}$$

$$\sigma_{33} = \lambda g^{ss}\left\{\nabla_s u_s + z\nabla_s u_s^{(1)} + \frac{1}{2} z^2\nabla_s u_s^{(2)} + \dots\right\} +$$
$$+ (\lambda + 2\mu)\left\{u_3^{(1)} + z u_3^{(2)} + \frac{1}{2} z^2 u_3^{(3)} + \dots\right\}$$
$$(i,\ s = 1,\ 2). \qquad (7.1b)$$

Equations (7.1a) and (7.1b) determine the stress tensor components displaced to the basic surface (cf.I. Sect.12). According to the Ricci theorem, (I, Sect.9), under this parallel displacement, the metric tensor g^z_{ik} is transformed into the metric tensor on the basic surface. For this reason, the quantities g^{ss} entering into eq.(7.1b) are contravariant components of the metric tensor on the basic surface.

To avoid misunderstandings, let us note the properties of the covariant derivatives of $\nabla_i u_j^{(n)}$ (i = 1, 2; j = 1, 2, 3). These derivatives, as before, are determined in three-dimensional space. In three-dimensional space, the functions of $u_j^{(n)}$ are components of a tensor of rank n + 1. This, according to eq.(I, 9.12), defines the meaning of the covariant derivatives $\nabla_i u_j^{(n)}$.

To determine the fundamental functions, we make use of the conditions on the boundary surfaces of the shell (Bibl.23b). On the boundary surfaces the components of the external forces are usually assigned.

To set forth the essence of the method, let us confine ourselves to the case of a shell of constant thickness and let us assume that the basic surface coincides with the middle surface of the shell. Consider the boundary conditions (II, 8.2b). Under the simplifying hypotheses adopted, these conditions will be of the following form:

$$\sigma_{i3}\big|_{z=h} \stackrel{*}{=} X_{(+)i}; \quad \sigma_{i3}\big|_{z=-h} \stackrel{*}{=} -X_{(-)i}. \qquad (7.2)$$

All the quantities entering into eqs.(7.2) are assumed to be displaced parallel to themselves on the basic surface, according to previous statements (I, Sect.11 and 12).

The series representing the components of the displacement and stresses are assumed to converge within the shell and on its surface*. Making use of eqs.(7.1a) - (7.1b), we obtain the following six equations: /97

$$\mu\left\{\nabla_i u_3 + u_i^{(1)} + h\,[\partial_i u_3^{(1)} + k_i(\nabla_i u_3 + u_i^{(1)}) + u_i^{(2)}] + \right.$$ (continued)

* This hypothesis is, as will be clear from the contents of Section 2, the most vulnerable point of the reduction method under consideration.

$$+\frac{1}{2}h^2(\nabla_i u_3^{(2)}+u_i^{(3)})+\ldots\Big\}=X_{(+)i}; \quad (7.3a)$$

$$\mu\Big\{\nabla_i u_3+u_i^{(1)}-h[\partial_i u_3^{(1)}+k_i(\nabla_i u_3+u_i^{(1)})+u_i^{(2)}]+$$
$$+\frac{1}{2}h^2(\nabla_i u_3^{(2)}+u_i^{(3)})-\ldots\Big\}=-X_{(-)i}; \quad (7.3b)$$

$$\lambda g^{ss}\Big\{\nabla_s u_s+h\nabla_s u_s^{(1)}+\frac{1}{2}h^2\nabla_s u_s^{(2)}+\ldots\Big\}+$$
$$+(\lambda+2\mu)\{u_3^{(1)}+hu_3^{(2)}+\frac{1}{2}h^2u_3^{(3)}+\ldots\Big\}=X_{(+)3}; \quad (7.3c)$$

$$\lambda g^{ss}\Big\{\nabla_s u_s-h\nabla_s u_s^{(1)}+\frac{1}{2}h^2\nabla_s u_s^{(2)}-\ldots\Big\}+(\lambda+2\mu)\times$$
$$\times\Big\{u_3^{(1)}-hu_3^{(2)}+\frac{1}{2}h^2u_3^{(3)}-\ldots\Big\}=-X_{(-)3}$$
$$(s=1,\ 2). \quad (7.3d)$$

This system may be replaced by its equivalent:

$$\sum_{m=0}^{\infty}\frac{1}{(2m)!}h^{2m}[\nabla_i u_3^{(2m)}+u_i^{(2m+1)}]=\frac{X_{(+)i}-X_{(-)i}}{2\mu}; \quad (7.4a)$$

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}h^{2m}\{\partial_i u_3^{(2m+1)}+k_i[\nabla_i u_3^{(2m)}+u_i^{(2m+1)}]+u_i^{(2m+2)}\}=$$
$$=\frac{X_{(+)i}+X_{(-)i}}{2h\mu}; \quad (7.4b)$$

$$\sum_{m=0}^{\infty}\frac{1}{(2m)!}h^{2m}\{\lambda g^{ss}\nabla_s u_s^{(2m)}+(\lambda+2\mu)u_3^{(2m+1)}\}=\frac{X_{(+)3}-X_{(-)3}}{2}; \quad (7.4c)$$

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}h^{2m}\{\lambda g^{ss}\nabla_s u_s^{(2m+1)}+(\lambda+2\mu)u_3^{(2m+2)}\}=\frac{X_{(+)3}+X_{(-)3}}{2h}$$
$$(i,\ s=1,\ 2). \quad (7.4d)$$

To these equations we must associate the relations (6.4a) - (6.4b). /98

Eliminating from eqs.(7.4a) - (7.4b) on the basis of eqs.(6.4a) - (6.4b) the quantities $u_j^{(n)}$ (f = 1, 2, 3; n = 2, 3...), we obtain a system of six equations in six unknown functions of u_j and $u_j^{(1)}$ (j = 1, 2, 3). This system will be of an order depending on the number of terms in the expansions. In turn, the number of terms in the expansions will depend on the prescribed error of the wanted result. Consequently, the order of the system of equations set up

by us may be very high*.

Let us return to eqs.(7.1a) - (7.1b). Making use of eqs.(7.4a) - (7.4d), we find

$$\sigma_{i3} = \frac{X_{(+)i} - X_{(-)i}}{2} + \frac{X_{(+)i} + X_{(-)i}}{2} \frac{z}{h} + \\ + \frac{\mu}{2}(z^2 - h^2)[\nabla_i u_3^{(2)} + u_i^{(3)}] + \ldots; \tag{7.5a}$$

$$\sigma_{33} = \frac{X_{(+)3} - X_{(-)3}}{2} + \frac{X_{(+)3} + X_{(-)3}}{2} \frac{z}{h} + \\ + \frac{1}{2}(z^2 - h^2)[\lambda g^{ss} \nabla_s u_s^{(2)} + 2\mu u_3^{(3)}] + \ldots \\ (i = 1,\ 2). \tag{7.5b}$$

Equations (7.5a) - (7.5b) establish the law of distribution of the normal part of the stress tensor over the thickness of the shell. These equations hold for the linear problems of the statics and dynamics of shells**.

Section 8. Further Development of the Classification of Shells with Respect to Dynamic Problems

Let us return to the classification of shells into "thin" and "non-thin". As will be clear from Sect.3, in the shell theory the quantity 2h is usually considered a small quantity if the natural unit of length is taken as one of the characteristic dimensions of the shell. In the problems of dynamics, such an approach to the classification of shells is inadequately motivated.

An analysis of the question of the applicability limits of the equations of the classical theory of flexure of plates to the solution of the dynamic /99
problems was performed by G.I.Petrashen (Bibl.30)***. Although this work relates to a special kind of shell, its results permit general conclusions that

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* This is also clear from a study of the statically stressed and strained states of an elastic medium by the method under consideration (cf. Bibl.96).

** Equations analogous to eqs.(7.5a) - (7.5b) are presented by us elsewhere (Bibl.23b). Analogous relations were given subsequently by (Bibl.16, Bibl.26), and others.

*** The further development of the investigation by Petrashen is contained in the paper by L.A.Molotkov "Engineering Functions for the Vibrations of Plates with Layered Structure", Leningrad Sect. Inst.Mat. USSR Academy of Sciences, Coll. V: "Questions of Dynamic Theory of the Propagation of Seismic Waves", 1961.

are valid for more general problems of the dynamics of shells. We shall therefore briefly enumerate his conclusions (Bibl.30).

This study was based on a solution, exact within the limits of the linear theory of elasticity, of the problem of the vibrations of an unbounded elastic layer under the action of a plane and axisymmetric surface load, and also that of a normal load uniformly distributed over a cross section of the surface layer. On the basis of an analysis of the solutions obtained, Petrashen came to the conclusion that the thickness of a plate for which the application of the theory of thin plates was still possible, depends substantially on the properties of the force influencing the plate.

In the first place, the width of the application zone of the load and the zone of its appreciable variation must considerably exceed the thickness of the plate, and in the second place the load must vary slowly. The latter requirement may be represented by the inequality

$$T \geqslant N\tau. \tag{8.1}$$

where T is the duration of appreciable variation of the surface load, N is a large number, and

$$\tau = 2hb, \tag{8.2}$$

where b^{-1} is the velocity of propagation of transverse elastic waves. Consequently, τ is the duration of the passage of the elastic transverse wave through the section of the layer.

We may also note the relation pointed out by Petrashen between the regions of the low-frequency spectrum ν and the thickness of the shell. This relation is of the form:

$$\nu bh \ll 1. \tag{8.3}$$

Thus, by increasing the thickness of the shell we decrease the region of frequencies ν in which the approximate theory of plates does not lead to considerable errors. It follows from Petrashen's study (Bibl.30) in particular, that the natural unit of length that can be adopted is the quantity $\nu^{-1}b^{-1}$, as will be clear from eq.(8.3).

The corresponding length of the sine wave, as is generally known, is expressed by the equation

$$l_* = 2\pi\nu^{-1}b^{-1}. \tag{8.4}$$

From this follows the possibility of choosing l_*, in solving certain problems of the elastodynamics of shells, as a natural unit of length. These facts, established by means of analyses of rigorous but partial solutions of boundary /100

dynamic problems of the theory of elasticity, can undoubtedly be extended with only minor changes to the shell theory. We do not, however, know of any general investigations that would permit introducing additional terms into the equations of the classical theory of plates and shells so as to make the solutions of the generalized equations represent, with sufficient accuracy, the solutions of the corresponding boundary conditions of elasticity*.

Apparently, the solution of the approximation equations can with sufficient accuracy describe only some part of the elastodynamic process studied, for example, some definite segment of the frequency spectrum, the phase or group velocity of waves with dispersion, etc. . For this reason, the division of shells into the classes "thin" and "non-thin" must be subordinated from the beginning to the problem of studying certain characteristics of the dynamic process. From this point of view, the conditions (8.1) - (8.3) determine the class of thin shells, depending on the desired accuracy of the study of the results of perturbing forces applied to them. In this connection we note that it is also possible in the problems of dynamics to introduce various natural units of length, subordinating them to the fundamental purpose of the subsequent investigation.

For instance, let us propose to study the propagation of elastic waves of lengths not less than l_* in an unbounded shell, i.e., in a shell homeomorphous with an unbounded layer. Let us put, according to eqs.(3.1) and (8.4),

$$\max\left(\frac{2h}{l_*}\right) = \eta. \tag{8.5}$$

Let N be the number of first terms retained in the above-discussed expansions and ϵ the prescribed relative deviation of some characteristic quantity (for example, of the phase or group velocity of waves with dispersion), determined - on the basis of the approximate theory of shells - from the value of this quantity derived from the equations of the three-dimensional problem. Then, it is possible to find

$$\eta = \eta(N, \varepsilon), \tag{8.6}$$

where the condition (8.5) will define the class of thin shells.

For all types of waves of length satisfying the inequality

$$l \geqslant l_*, \tag{8.7}$$

the shell will likewise be thin. For other waves, the shell will not be thin, i.e., the number of terms retained will not ensure the necessary accuracy of

* The equation obtained by Petrashen (Bibl.30) on the basis of the solutions of the above-mentioned partial problems of the elasticity theory give no general answer to the question posed.

solution.

We recall that the study of another report (Bibl.30) was based on an analysis of rigorous solutions for an unbounded elastic layer. Consequently, the definition of thin shells indicated here may require substantial additions, or even be unsuitable for solving dynamic boundary problems in the case of bounded shells. /101

The above statements and those in Sect.3 lead to the conclusion that there exists no general criterion that would permit a classification of shells into these classes. There is also no general natural unit of length resulting from the properties of the dynamic processes in shells. In concrete problems, however, the introduction of a suitably chosen natural unit of length may prove useful. We shall assume below that such a physical or geometrical unit has been selected and that the quantity 2h is sufficiently small, i.e., that the conditions (3.1) are satisfied, with the possible replacement of the second condition by the relation (8.3) or (8.4). The use of the conditions (3.3) and (3.4) is likewise possible.

Section 9. Method of Successive Approximations

Although the system of equations (7.4a) - (7.4d) constitutes the foundation of one of the possible analytic statements of the problem of reduction, the complexity of this system and the absence of a criterion allowing preliminary conclusions as to convergence of the series on the left-hand side of these equations forces us to turn to methods that permit solution of the problem of reduction without integrating eqs.(7.4a) - (7.4d). Such a method has been given by us elsewhere (Bibl.23a, b). It is the method of successive approximations, based on the hypothesis that 2h is relatively small.

To develop the process of successive approximations, let us make use of eqs.(7.4a) and (7.4c). Equations (7.4b) and (7.4d) will not as yet be applied. Subsequently, eqs.(7.4b) and (7.4d) will permit us to develop one of the alternative versions of the solution of the reduction problem. If we do not use these equations, then auxiliary equations predetermining the statement of the problem must be set up.

To find the first (initial) approximation, let us reject from eqs.(7.4a) and (7.4c) all terms containing h. Then,

$$[u_i^{(1)}]_1 \overset{*}{=} \frac{X_{(+)i} - X_{(-)i}}{2\mu} - \nabla_i u_3, \tag{9.1}$$

$$[u_3^{(1)}]_1 \overset{*}{=} \frac{X_{(+)3} - X_{(-)3}}{2(\lambda + 2\mu)} - \frac{\lambda}{\lambda + 2\mu} g^{ss} \nabla_s u_s \qquad (i,\ s = 1,\ 2). \tag{9.2}$$

To find the next approximation, let us make use of eqs.(6.4a) - (6.4b), putting n = 2 and n = 3. Determining the first approximation for $[u_i^{(n)}]_1$, and /102

$[u_3^{(n)}]_1$, we return to eqs.(7.4a) and (7.4c). From these equations, retaining in them all terms with the factor h^2, we will find the second approximation of the basic quantities:

$$[u_i^{(1)}]_2 \stackrel{*}{=} \frac{X_{(+)i} - X_{(-)i}}{2\mu} - \nabla_i u_3 - \frac{1}{2} h^2 \{\nabla_i [u_3^{(2)}]_1 + [u_i^{(3)}]_1\}, \tag{9.3}$$

$$[u_3^{(1)}]_2 \stackrel{*}{=} \frac{X_{(+)3} - X_{(-)3}}{2(\lambda + 2\mu)} - \frac{\lambda}{\lambda + 2\mu} g^{ss} \nabla_s u_s - \frac{1}{2} h^2 \times$$
$$\times \left\{ \frac{\lambda}{\lambda + 2\mu} g^{ss} \nabla_s [u_s^{(2)}]_1 + [u_3^{(3)}]_1 \right\} \qquad (i,\ s = 1,\ 2). \tag{9.4}$$

The process can be continued further. Applying this method let us find the n-th approximation for the components $\varepsilon_{i3}^{(0)}$. These quantities will hereafter be called $[\varepsilon_{i3}]_n$ (i = 1, 2, 3). We have:

$$2[\varepsilon_{i3}]_n = \nabla_i u_3 + [u_i^{(1)}]_n \qquad (i = 1,\ 2); \tag{9.5a}$$

$$[\varepsilon_{33}]_n = [u_3^{(1)}]_n. \tag{9.5b}$$

We find:

$$[u_i^{(1)}]_n = 2[\varepsilon_{i3}]_n - \nabla_i u_3 \qquad (i = 1,\ 2); \tag{9.6a}$$

$$[u_3^{(1)}]_n = [\varepsilon_{33}]_n. \tag{9.6b}$$

Equations (9.6a) - (9.6b) permit us to derive formulas reflecting, in explicit form, the deviation of the proposed shell theory from the classical theory*. Equations (9.1) - (9.6b) will hereafter be called the reduction formulas:

Let us make a preliminary analysis of the relations obtained.

1. If there is no load on the boundary surfaces of the shell, then from eqs.(9.1) - (9.2) follow the relations:

$$[u_i^{(1)}]_1 \stackrel{*}{=} -\nabla_i u_3; \quad [u_3^{(1)}]_1 \stackrel{*}{=} -\frac{\lambda}{\lambda + 2\mu} g^{ss} \nabla_s u_s. \tag{9.7}$$

These equations express the condition of invariance of an element of the basic surface normal to the middle surface. Thus the first approximation is close to the Kirchhoff-Love hypothesis, which is still less restrictive for the

* Here we have somewhat modified the notation adopted by us elsewhere(Bibl.23b).

strains than this hypothesis.

2. The method of successive approximation given here requires the differentiability of the functions $X_{(\pm)j}$ and ρF_j.

3. We note the rules for covariant differentiation of the functions $X^j_{(\pm)}$. Although these functions are essentially components of a contravariant vector, they nevertheless express, according to the reduction formulas, the components of a covariant tensor of second rank. This determines the rules for their covariant differentiation. /103

4. We have not been able to establish a general proof of the convergence of the process of successive approximation suggested by us. Elsewhere (Bibl.23b) we have indicated methods for the preliminary approach to such a proof in the case of static problems. As for the problems of dynamics, the difficulties here are still considerable. The question of the convergence of the process of successive approximations may be approached in the problems of statics based on general analytic properties of the solutions of the boundary problems of the elasticity theory mentioned in Sect.2. It can be asserted that, for the cases of the action of forces determined by functions of a point without analytic singularities, the series constructed by us will in fact converge. But these series will apparently diverge in the neighborhood of the points of application of concentrated forces. Of course, concentrated forces are one of the forms of limiting abstractions. It is clear that even here we can obtain a solution that is satisfactory from the physical viewpoint by substituting the concentrated force by its equivalent load, distributed over a small but finite region of the body.

The analytic properties of the solutions of dynamic problems of the elasticity theory as investigated to date, do not permit a definite answer to the question whether the successive approximations developed by our method actually converge*. For this reason, we must consider the proposed method as merely an algorithm for obtaining approximate equations of the dynamics of elastic shells. These equations are subject to further experimental and theoretical verification.

There are indirect confirmations of our methods. For example, in the work by M.P.Petrenko and that by I.T.Selezov** it is shown that the method under discussion permits obtaining equations for the longitudinal and transverse vibrations of rods and the transverse vibrations of plates which yield, as special cases, the equations found by other methods and by other authors. In this manner, it is possible to obtain a generalization of the differential equation

* The state of the general theory of solution of the problems of elastodynamics is indicated in V.D.Kupradze's book "Boundary Problems of the Theory of Vibrations and Integral Equations", Gostekhizdat, 1950.

** Cf. the dissertation by M.P.Petrenko "Longitudinal and Transverse Vibrations Arising in Short Rods of Constant and Variable Thickness under the Action of an Impact" Institute of Mechanics, Ukr SSR Academy of Sciences, 1961, and the above-cited dissertation by I.T.Selezov.

of the longitudinal vibrations of rods found by S.P.Timoshenko, as well as various generalizations of known equations for transverse vibrations of plates, for example the equations given by Ya.S.Uflyand, et al. These results appar- /104 ently confirm the expedience of the method proposed here.

Returning to the question of the convergence of the above-suggested method of successive approximations, it is useful to cite the concepts by A.N.Krylov on the convergence in purely analytical and applied research*. Obviously, "convergence" is important here in view of the fact that, after a finitely small number of approximations, it will yield sufficiently exact equations, i.e., equations whose solutions will satisfy the equations of the mathematical theory of elasticity and the boundary conditions, with an error sufficiently small from the viewpoint of practical requirements.

The study (Bibl.30) on the classical theory of plates shows that the above-mentioned "practical convergence" will take place whenever the restrictions indicated in Sect.8 are imposed on the acting forces. The question of the applicability limits of the equations obtained by this method requires further investigation.

Section 10. Expansion of the Strain Tensor into a Tangential Part and a Normal Part

Let us return to the expansions (4.5a) - (4.5c) of the components of the strain tensor $\epsilon_{ik}^{(z)}$ (i, k = 1, 2). These components were not used by us in solving the reduction problems. As we shall show, for z = 0 they describe the deformation of the basic surface, i.e., they determine, with an accuracy to quantities of the second order of smallness, the variations of the metric tensor components of the basic surface. They also contain terms depending on the variation of the curvature of the basic surface.

We will denote the set of terms of $\epsilon_{ik}^{(z)}$ determining the variation of the metric tensor components of the basic surface, as the tangential part of the strain tensor. We shall call the set of quantities belonging to $\epsilon_{ik}^{(z)}$ and depending on the curvature variation of the basic surface, the normal part of the strain tensor. Let us represent eqs.(4.5a) - (4.5c) in the following form:

$$\epsilon_{11}^{(z)} = \epsilon_{11} - z\varkappa_{11} + \ldots, \tag{10.1a}$$

$$\epsilon_{12}^{(z)} = \epsilon_{12} - z\varkappa_{12} + \ldots, \tag{10.1b}$$

$$\epsilon_{22}^{(z)} = \epsilon_{22} - z\varkappa_{22} + \ldots \quad . \tag{10.1c}$$

The quantities ϵ_{ik} form the tangential part of the strain tensor. These /105 quantities are determined by the formulas:

$$\epsilon_{11} = \partial_1 u_1 - \frac{1}{2g_{11}} u_1 \partial_1 g_{11} + \frac{1}{2g_{22}} u_2 \partial_2 g_{11} - k_1 g_{11} u_3, \tag{10.2a}$$

* Cf. "Collection of Works of Academician A.N.Krylov", Vol.X, USSR Academy of Sciences, 1948, pp.205-206.

$$\varepsilon_{22} = \partial_2 u_2 + \frac{1}{2g_{11}} u_1 \partial_1 g_{22} - \frac{1}{2g_{22}} u_2 \partial_2 g_{22} - k_2 g_{22} u_3, \qquad (10.2^{b}) \qquad (10.2b)$$

$$\varepsilon_{12} = \partial_2 u_1 + \partial_2 u_2 - \frac{u_2}{g_{11}} \partial_1 g_{11} - \frac{u_1}{g_{22}} \partial_1 g_{22}. \qquad (10.2^{c}) \qquad (10.2c)$$

The quantities $\varkappa_{ik}$ are the components of the symmetric covariant tensor of second rank on the basic surface. This is known as the tensor of variations of curvature. The connection between the quantities $\varkappa_{ik}$ and the variations of curvature will be clear from eq.(4.2), bearing in mind that the derivatives ∇_3 are absolute derivatives in the direction of a normal to the basic surface and making use of eqs.(I, 3.6a) - (I, 3.6b). The quantities $\varkappa_{ik}$, corresponding to the n-th approximation, are determined on the basis of eqs.(4.5a) - (4.5c) and the reduction formulas (9.6a) - (9.6b):

$$\begin{aligned}[\varkappa_{11}]_n = {} & \partial_1(\partial_1 u_3 + k_1 u_1) - k_1\varepsilon_{11} - \frac{1}{2g_{11}} \partial_1 g_{11}(\partial_1 u_3 + k_1 u_1) + \\ & + \frac{1}{2g_{22}} \partial_2 g_{11}(\partial_2 u_3 + k_2 u_2) - 2\partial_1[\varepsilon_{13}]_n + \frac{1}{g_{11}} \partial_1 g_{11} [\varepsilon_{13}]_n - \\ & - \frac{1}{g_{22}} \partial_2 g_{11} [\varepsilon_{23}]_n + g_{11} k_1 [\varepsilon_{33}]_n, \end{aligned} \qquad (10.3a)$$

$$\begin{aligned}[\varkappa_{22}]_n = {} & \partial_2(\partial_2 u_3 + k_2 u_2) - k_2\varepsilon_{22} + \frac{1}{2g_{11}} \partial_1 g_{22}(\partial_1 u_3 + k_1 u_1) - \\ & - \frac{1}{2g_{22}} \partial_2 g_{22}(\partial_2 u_3 + k_2 u_2) - 2\partial_2[\varepsilon_{23}]_n - \frac{1}{g_{11}} \partial_1 g_{22} [\varepsilon_{13}]_n + \\ & + \frac{1}{g_{22}} \partial_2 g_{22} [\varepsilon_{23}]_n + g_{22} k_2 [\varepsilon_{33}]_n, \end{aligned} \qquad (10.3b)$$

$$\begin{aligned}2[\varkappa_{12}]_n = {} & \partial_1(\partial_2 u_3 + k_2 u_2) + \partial_2(\partial_1 u_3 + k_1 u_1) - k_1 \nabla_1 u_2 - k_2 \nabla_2 u_1 - \\ & - \frac{1}{g_{11}} \partial_2 g_{11}(\partial_1 u_3 + k_1 u_1) - \frac{1}{g_{22}} \partial_1 g_{22}(\partial_2 u_3 + k_2 u_2) - \\ & - 2\partial_1[\varepsilon_{23}]_n - 2\partial_2[\varepsilon_{13}]_n + \frac{2}{g_{11}} \partial_2 g_{11} [\varepsilon_{13}]_n + \frac{2}{g_{22}} \partial_1 g_{22} [\varepsilon_{23}]_n. \end{aligned} \qquad (10.3c)$$

The terms containing $[\varepsilon_{ik}]_n$ are absent from the equations of the classical theory. They obviously characterize the influence of the local loads on the curvature of the shell.

The relations (10.2a) - (10.3c) constitute the first (kinematic) group /106

of equations of the shell theory set up on the classical plane. The terms of the expansions of $\epsilon_{ik}^{(z)}$ containing z^n, where $n \geq 2$, have no special names and their geometrical meaning is more complicated than that of ϵ_{ik} and $\varkappa_{ik}$. The scope of the present study does not permit us to go more deeply into these kinematic investigations.

Section 11. Two Methods of Setting up the Equations of the Theory of Shells, both Connected with the Method of Successive Approximations. First Version of Establishment of the Elastodynamic Systems of Equations

The method of successive approximations requires the use of three equations derived from the system of six equations (7.4a) - (7.4d). In essence, this method is one of the methods of eliminating the three unknown functions $u_i^{(1)}$ ($i = 1, 2, 3$) from the six unknowns in the system of equations (7.4a) - (7.4d). To obtain a complete system of equations of the shell theory, three more equations must be set up with unknown functions u_i ($i = 1, 2, 3$). This may be done by two methods.

The first method is based on the use of the three equations of the system (7.4a) - (7.4d) that had not been used by us in deriving the reduction formulas. By eliminating the quantities $u_i^{(1)}$ from these equations on the basis of the reduction formulas, we obtain a system of three equations with unknown functions u_i ($i = 1, 2, 3$).

The second method consists in setting up the conditions of equilibrium of an element of the shell as a whole, followed by the application of relations resulting from Hooke's law, and is the most widely used in modern shell theory, and essentially corresponds to the construction of the classical theory.

Consider the elastodynamic system of equations of the theory of shells, derived from eqs.(7.4a) - (7.4d) and from the reduction formulas, and let us retain the first version. First we must establish the relative accuracy of the required system of equations. We shall conditionally define this accuracy by the highest power of h in the terms retained in the equations.

The equations of the classical theory of shells were usually confined to terms containing h^3, but introduced only some of these terms*. During the last 10 or 12 years, a number of studies on the dynamics of plates and cylindrical shells have been published, containing terms in h^3 but also omitting a number of terms of this order, without giving sufficient reasons for the legitimacy of neglecting them. Below, we also confine ourselves to setting up the equations of shell dynamics, containing all terms up to and including the factor h^3. /107
However, the method employed by us makes it possible to set up equations containing all terms to an arbitrary power of h**.

* We have given elsewhere (Bibl.23b) a detailed analysis of the completeness of the classical system of equations of the statics of shells.

** In the above-cited dissertation by I.T.Selezov, the equations of the vibrations of plates, containing terms up and including h^5, were set up by this method in the expanded form.

To obtain the required degree of accuracy, the two first terms and the series must be retained in the left-hand sides of eqs.(7.4b) - (7.4d). Then we obtain the following approximation formulas:

$$\nabla_i u_3^{(1)} + u_i^{(2)} + \frac{1}{6} h^2 (\nabla_i u_3^{(3)} + u_i^{(4)}) = \frac{X_{(+)i} + X_{(-)i}}{2h\mu}, \tag{11.1a}$$

$$\lambda g^{ss} \nabla_s u_s^{(1)} + (\lambda + 2\mu) u_3^{(2)} + \frac{1}{6} h^2 \{\lambda g^{ss} \nabla_s u_s^{(3)} + (\lambda + 2\mu) u_3^{(4)}\} =$$

$$= \frac{X_{(+)3} + X_{(-)3}}{2h} \qquad (i, s = 1, 2). \tag{11.1b}$$

Let us make use of the recurrent relations (6.4a) - (6.4b) and the reduction formulas, and let us now make all the calculations with the required degree of detail. On the basis of eqs.(6.4a) - (6.4b) we find successively

$$u_i^{(2)} = L_i u_3^{(1)} + L_{i.}^{\cdot s} u_s + M u_i - \frac{\rho}{\mu} F_i; \tag{11.2a}$$

$$u_3^{(2)} = N_{3.}^{\cdot s} u_s^{(1)} + \frac{\mu}{\lambda + 2\mu} M u_3 - \frac{\rho}{\lambda + 2\mu} F_3; \tag{11.2b}$$

$$u_i^{(3)} = L_i u_3^{(2)} + L_{i.}^{\cdot s} u_s^{(1)} + M u_i^{(1)} - \frac{\rho}{\mu} F_i^{(1)} = (L_i N_{3.}^{\cdot s} + L_{i.}^{\cdot s}) u_s^{(1)} +$$

$$+ M \left(u_i^{(1)} + \frac{\mu}{\lambda + 2\mu} L_i u_3\right) - \frac{\rho}{\lambda + 2\mu} L_i F_3 - \frac{\rho}{\mu} F_i^{(1)}; \tag{11.3a}$$

$$u_3^{(3)} = N_{3.}^{\cdot s} u_s^{(2)} + \frac{\mu}{\lambda + 2\mu} M u_3^{(1)} - \frac{\rho}{\lambda + 2\mu} F_3^{(1)} =$$

$$= \left(N_{3.}^{\cdot s} L_s + \frac{\mu}{\lambda + 2\mu} M\right) u_3^{(1)} + (N_{3.}^{\cdot s} L_{s.}^{\cdot r} + N_{3.}^{\cdot r} M) u_r -$$

$$- \frac{\rho}{\mu} N_{3.}^{\cdot s} F_s - \frac{\rho}{\lambda + 2\mu} F_3^{(1)}; \tag{11.3b}$$

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$$u_i^{(4)} = L_i u_3^{(3)} + L_{i.}^{\cdot s} u_s^{(2)} + M u_i^{(2)} - \frac{\rho}{\mu} F_i^{(2)} =$$

$$= \left(L_i N_{3.}^{\cdot s} L_s + L_{i.}^{\cdot s} L_s + \frac{\lambda + 3\mu}{\lambda + 2\mu} L_i M\right) u_3^{(1)} +$$

$$+ [N_{3.}^{\cdot s} L_{s.}^{\cdot r} L_i + (N_{3.}^{\cdot r} L_i + 2L_{i.}^{\cdot r}) M + L_{i.}^{\cdot s} L_{s.}^{\cdot r}] u_r + M^2 u_i -$$

$$- \frac{\rho}{\mu} \{[L_i N_{3.}^{\cdot s} + L_{i.}^{\cdot s}] F_s + M F_i + F_i^{(2)}\} - \frac{\rho}{\lambda + 2\mu} L_i F_3^{(1)}; \tag{11.4a}$$

$$u_3^{(4)} = N_{3.}^{.s} u_s^{(3)} + \frac{\mu}{\lambda+2\mu} M u_3^{(2)} - \frac{\rho}{\lambda+2\mu} F_3^{(2)} =$$

$$= \left(N_{3.}^{.s} L_s N_{3.}^{.r} + N_{3.}^{.s} L_{s.}^{.r} + \frac{\lambda+3\mu}{\lambda+2\mu} M N_{3.}^{.r} \right) u_r^{(1)} +$$

$$+ \frac{\mu}{\lambda+2\mu} M \left(N_{3.}^{.s} L_s + \frac{\mu}{\lambda+2\mu} M \right) u_3 - \frac{\rho}{\mu} N_{3.}^{.s} F_s^{(1)} -$$

$$- \frac{\rho}{\lambda+2\mu} \left(N_{3.}^{.s} L_s + \frac{\mu}{\lambda+2\mu} M \right) F_3 - \frac{\rho}{\lambda+2\mu} F_3^{(2)}$$

$$(i, r, s = 1, 2). \qquad (11.4b)$$

Now, applying the reduction formulas, we can exclude the quantities $u_j^{(1)}$ from the resultant equation. Here, in view of the prescribed arbitrary accuracy of the equations, we will introduce the first approximations in the expressions $u_k^{(3)}$ and $u_j^{(4)}$, while the remaining quantities entering into eqs.(11.1a) - (11.1b) can be determined by the second approximations.

In order to execute this program, we must return to eqs.(9.1) - (9.2) and introduce there our newly adopted notation, and then find the first approximation of the quantities $u_j^{(2)}$, $u_j^{(3)}$, $u_j^{(4)}$. Then we will be able to write eqs.(9.3) - (9.4) in the expanded form and complete setting up the system of equations (11.1a) - (11.1b). From eqs.(6.3a) - (6.3b), we find

$$\nabla_i = -\frac{\mu}{\lambda+\mu} L_i; \quad g^{ss} \nabla_s = -\frac{\lambda+2\mu}{\lambda+\mu} N_{3.}^{.s}. \qquad (11.5)$$

We put

$$Y_{i3} \overset{*}{=} \frac{X_{(+)i} - X_{(-)i}}{2\mu}; \quad Y_{33} \overset{*}{=} \frac{X_{(+)3} - X_{(-)3}}{2(\lambda+2\mu)}. \qquad (11.6)$$

We have here considered the remarks in Sect.9 on the meaning of the functions $X_{(\pm)i}$. Equations (9.1) - (9.2) then take the following form:

$$[u_i^{(1)}]_1 \overset{*}{=} Y_{i3} + \frac{\mu}{\lambda+\mu} L_i u_3, \qquad (11.7a)$$

$$[u_3^{(1)}]_1 \overset{*}{=} Y_{33} + \frac{\lambda}{\lambda+\mu} N_{3.}^{.s} u_s. \qquad (11.7b)$$

Further elementary but unwieldy calculations lead to the following general expressions of the wanted quantities: /109

$$[u_i^{(2n)}]_m = [P^{(2n)}]_m u_i + [R_i^{(2n)\,s}]_m u_s + [Q_i^{(2n)}]_m; \qquad (11.8a)$$

$$[u_3^{(2n)}]_m = [S_3^{(2n)}]_m\, u_3 + [Q_3^{(2n)}]_m; \tag{11.8b}$$

$$[u_i^{(2n-1)}]_m = [S_i^{(2n-1)}]_m u_3 + [Q_i^{(2n-1)}]_m; \tag{11.9a}$$

$$[u_3^{(2n-1)}]_m = [R_3^{(2n-1)s}]_m u_s + [Q_3^{(2n-1)}]_m \tag{11.9b}$$

$$(i, s = 1, 2).$$

The expressions $[P^{(2n)}]_m$, $[R_i^{(2n)s}]_m$, $[S_3^{(2n)}]_m$, $[S_i^{(2n-1)}]_m$, $[R_3^{(2n-1)s}]_m$ are differential operators depending on the order of approximation. We shall indicate below the form of these operators of the approximations introduced by us. The quantities $[Q_j^{(k)}]_m$ are "force terms" containing the differential operations on the body forces and surface forces.

To start with, we indicate the values of the operators in eqs.(11.8a) - (11.9b) for the first approximation. We have

$$[P^{(2)}]_1 = M; \quad [R_i^{(2)s}]_1 = \frac{\lambda}{\lambda+\mu} L_i N_{3.}^{;s} + L_{i.}^{;s}; \tag{11.10a}$$

$$[S_3^{(2)}]_1 = \frac{\mu}{\lambda+\mu} L_s N_{3.}^{;s} + \frac{\mu}{\lambda+2\mu} M; \tag{11.10b}$$

$$[S_i^{(3)}]_1 = \frac{\mu}{\lambda+\mu} [(L_i N_{3.}^{;s} + L_{i.}^{;s}) L_s + 2ML_i]; \tag{11.11a}$$

$$[R_3^{(3)s}]_1 = \frac{\lambda}{\lambda+\mu} L_r N_{3.}^{;r} N_{3.}^{;s} + \frac{\lambda^2 + 4\mu\lambda + 2\mu^2}{(\lambda+\mu)(\lambda+2\mu)} M N_{3.}^{;s} + N_{3.}^{;r} L_{r.}^{;s}; \tag{11.11b}$$

$$[P^{(4)}]_1 = M^2; \quad [R_i^{(4)s}]_1 = \frac{\lambda}{\lambda+\mu}\Big(L_i L_r N_{3.}^{;r} N_{3.}^{;s} + L_{i.}^{;r} L_r N_{3.}^{;s} +$$

$$+ \frac{\lambda+3\mu}{\lambda+2\mu} M L_i N_{3.}^{;s}\Big) + N_{3.}^{;r} L_{r.}^{;s} L_i + (N_{3.}^{;s} L_i + 2L_{i.}^{;s}) M + L_{i.}^{;r} L_{r.}^{;s}; \tag{11.12a}$$

$$[S_3^{(4)}]_1 = \frac{\mu}{\lambda+\mu} (L_s L_r N_{3.}^{;r} N_{3.}^{;s} + L_s L_{r.}^{;s} N_{3.}^{;r} + 2M L_s N_{3.}^{;s}) +$$

$$+ \left(\frac{\mu}{\lambda+2\mu}\right)^2 M^2 \qquad (i, r, s = 1, 2). \tag{11.12b}$$

The "force" operators have the following meaning:

$$[Q_i^{(2)}]_1 = L_i Y_{33} - \frac{\rho}{\mu} F_i; \tag{11.13a}$$

$$[Q_3^{(2)}]_1 = N_{3.}^{;s} Y_{s3} - \frac{\rho}{\lambda+2\mu} F_3; \tag{11.13b}$$

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$$[Q_i^{(3)}]_1 = MY_{i3} + (L_i N_{3.}^{;s} + L_{i.}^{;s})\, Y_{s3} - \frac{\rho}{\lambda+2\mu} L_i F_3 - \frac{\rho}{\mu} F_i^{(1)}; \tag{11.14a}$$

$$[Q_3^{(3)}]_1 = \left(N_{3.}^{;s} L_s + \frac{\mu}{\lambda+2\mu} M\right) Y_{33} - \frac{\rho}{\mu} N_{3.}^{;s} F_s - \frac{\rho}{\lambda+2\mu} F_3^{(1)}; \tag{11.14b}$$

$$[Q_i^{(4)}]_1 = \left(L_i L_s N_{3.}^{;s} + L_s L_{i.}^{;s} + \frac{\lambda+3\mu}{\lambda+2\mu} ML_i\right) Y_{33} -$$
$$- \frac{\rho}{\mu} [(L_i N_{3.}^{;s} + L_{i.}^{;s})\, F_s + MF_i + F_i^{(2)}] - \frac{\rho}{\lambda+2\mu} L_i F_3^{(1)}; \tag{11.14c}$$

$$[Q_3^{(4)}]_1 = \left(L_r N_{3.}^{;r} N_{3.}^{;s} + N_{3.}^{;r} L_{r.}^{;s} + \frac{\lambda+3\mu}{\lambda+2\mu} M N_{3.}^{;s}\right) Y_{s3} - \frac{\rho}{\mu} N_{3.}^{;s} F_s^{(1)} -$$
$$- \frac{\rho}{\lambda+2\mu} \left(N_{3.}^{;s} L_s + \frac{\mu}{\lambda+2\mu} M\right) F_3 - \frac{\rho}{\lambda+2\mu} F_3^{(2)}. \tag{11.14d}$$

Equations (11.10a) - (11.14d), together with the relations (11.8a) - (11.9b), determine the first approximation.

Let us now consider the second approximation. To set up eqs.(11.1a) - (11.1b) with the necessary accuracy, it is sufficient to consider the second approximations of the quantities $u_j^{(1)}$ and $u_j^{(2)}$ (j = 1, 2, 3). Again starting from eqs.(11.8a) - (11.9b), we shall give the values of the differential and "force" operators contained in the expressions for $[u_j^{(1)}]_2$ and $[u_j^{(2)}]_2$. We have

$$[S_i^{(1)}]_2 = \frac{\mu}{\lambda+\mu} L_i - \frac{\mu h^2}{2(\lambda+\mu)} \left[\lambda L_i L_s N_{3.}^{;s} + L_s L_{i.}^{;s} + \frac{2(\lambda+\mu)}{\lambda+2\mu} ML_i\right]; \tag{11.15a}$$

$$[R_3^{(1)s}]_2 = \frac{\lambda}{\lambda+\mu} N_{3.}^{;s} - \frac{1}{2} h^2 \left[\frac{\mu}{\lambda+\mu} (L_r N_{3.}^{;r} N_{3.}^{;s} + L_{r.}^{;s} N_{3.}^{;r}) + \right.$$
$$\left. + \frac{2\mu}{\lambda+2\mu} MN_{3.}^{;s}\right]; \tag{11.15b}$$

$$[P^{(2)}]_2 = M; \quad [R_i^{(2)s}]_2 = L_{i.}^{;s} + \frac{\lambda}{\lambda+\mu} L_i N_{3.}^{;s} -$$
$$- \frac{1}{2} h^2 \left[\frac{\mu}{\lambda+\mu} (L_i L_r N_{3.}^{;r} N_{3.}^{;s} + L_i L_{r.}^{;s} N_{3.}^{;r} + \frac{2\mu}{\lambda+2\mu} ML_i N_{3.}^{;s}\right]; \tag{11.16a}$$

$$[S_3^{(2)}]_2 = \frac{\mu}{\lambda+2\mu} M + \frac{\mu}{\lambda+\mu} L_s N_{3.}^{.s} - \frac{\mu h^2}{2(\lambda+\mu)} \Big[\lambda L_s L_r N_{3.}^{.r} N_{3.}^{.s} + $$

$$+ L_r L_{s.}^{.r} N_{3.}^{.s} + \frac{2(\lambda+\mu)}{\lambda+2\mu} M L_s N_{3.}^{.s} \Big] \qquad (i, r, s=1, 2). \tag{11.16b}$$

Further, we find the "force" operators: /111

$$[Q_i^{(1)}]_2 = Y_{i3} - \frac{1}{2} h^2 \Big[\Big(L_{i.}^{.s} + \frac{\lambda}{\lambda+\mu} L_i N_{3.}^{.s} \Big) Y_{s3} + M Y_{i3} - \frac{\lambda\rho}{\lambda+2\mu} L_i F_3 -$$

$$- \frac{\rho}{\mu} F_i^{(1)} \Big]; \tag{11.17a}$$

$$[Q_3^{(1)}]_2 = Y_{33} - \frac{1}{2} h^2 \Big[\Big(\frac{\mu}{\lambda+\mu} N_{3.}^{.s} L_s + \frac{\mu}{\lambda+2\mu} M \Big) Y_{33} -$$

$$- \frac{\rho}{\lambda+\mu} N_{3.}^{.s} F_s - \frac{\rho}{\lambda+2\mu} F_3^{(1)} \Big]; \tag{11.17b}$$

$$[Q_i^{(2)}]_2 = L_i Y_{33} - \frac{\rho}{\mu} F_i - \frac{1}{2} h^2 \Big[\Big(\frac{\mu}{\lambda+\mu} L_i L_s N_{3.}^{.s} + \frac{\mu}{\lambda+2\mu} M L_i \Big) Y_{33} -$$

$$- \frac{\rho}{\lambda+\mu} L_i N_{3.}^{.s} F_s - \frac{\rho}{\gamma+2\mu} L_i F_3^{(1)} \Big]; \tag{11.18a}$$

$$[Q_3^{(2)}]_2 = N_{3.}^{.s} Y_{s3} - \frac{\rho}{\lambda+2\mu} F_3 - \frac{1}{2} h^2 \Big[\Big(L_{s.}^{.r} N_{3.}^{.s} + \frac{\lambda}{\lambda+\mu} L_s N_{3.}^{.s} N_{3.}^{.r} \Big) Y_{r3} +$$

$$+ M N_{3.}^{.s} Y_{s3} - \frac{\lambda\rho}{\lambda+2\mu} L_s N_{3.}^{.s} F_3 - \frac{\rho}{\mu} N_{3.}^{.s} F_s^{(1)} \Big]$$

$$(i, r, s = 1, 2). \tag{11.18b}$$

Let us return to eqs.(11.1a) - (11.1b). Making use of the notation (11.5) we represent these equations in the following form:

$$[u_i^{(2)}]_2 - \frac{\mu}{\lambda+} L_i [u_3^{(1)}]_2 + \frac{1}{6} h^2 \Big\{ [u_i^{(4)}]_1 - \frac{\mu}{\lambda+\mu} L_i [u_3^{(3)}]_1 \Big\} = Z_{i3}; \tag{11.19a}$$

$$[u_3^{(2)}]_2 - \frac{\lambda}{\lambda+\mu} N_{3.}^{.s} [u_s^{(1)}]_2 + \frac{1}{6} h^2 \Big\{ [u_3^{(4)}]_1 - \frac{\lambda}{\lambda+\mu} N_{3.}^{.s} [u_s^{(3)}]_1 \Big\} = Z_{33}$$

$$(i, s = 1, 2). \tag{11.19b}$$

where

$$Z_{i3} \stackrel{*}{=} \frac{X_{(+)i} + X_{(-)i}}{2h\mu}; \quad Z_{33} \stackrel{*}{=} \frac{X_{(+)3} + X_{(-)3}}{2h(\lambda+2\mu)}. \tag{11.20}$$

Applying eqs.(11.8a) - (11.9b), we find

$$\left\{[P^{(2)}]_2 + \frac{1}{6} h^2 [P^{(4)}]_1\right\} u_i + \left\{[R_i^{(2)\,s}]_2 - \frac{\mu}{\lambda+\mu} L_i [R_3^{(1)\,s}]_2 + \right.$$
$$\left. + \frac{1}{6} h^2 \left[[R_i^{(4)\,s}]_1 - \frac{\mu}{\lambda+\mu} L_i [R_3^{(3)\,s}]_1\right]\right\} u_s + [Q_i^{(2)}]_2 -$$
$$- \frac{\mu}{\lambda+\mu} L_i [Q_3^{(1)}]_2 + \frac{1}{6} h^2 \left\{[Q_i^{(4)}]_1 - \frac{\mu}{\lambda+\mu} L_i [Q_3^{(3)}]_1\right\} - Z_{i3} = 0; \tag{11.21a}$$

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$$\left\{[S_3^{(2)}]_2 - \frac{\lambda}{\lambda+\mu} N_{3.}^{\cdot s} [S_s^{(1)}]_2 + \frac{1}{6} h^2 \left[[S_3^{(4)}]_1 - \frac{\lambda}{\lambda+\mu} N_{3.}^{\cdot s} [S_s^{(3)}]_1\right]\right\} u_3 +$$
$$+ [Q_3^{(2)}]_2 - \frac{\lambda}{\lambda+\mu} N_{3.}^{\cdot s} [Q_s^{(1)}]_2 + \frac{1}{6} h^2 \left\{[Q_3^{(4)}]_1 - \frac{\lambda}{\lambda+\mu} N_{3.}^{\cdot s} [Q_s^{(3)}]_1\right\} - Z_{33} = 0$$
$$(i,\ s = 1,\ 2) \tag{11.21b}$$

We shall now make several preliminary remarks on the system of equations (11.21a) - (11.21b).

1. The system of equations (11.21a) - (11.21b) is of the twelfth order. We recall that the order of the system of equations in the classical theory is eight. The increase in the order of the system is due to the introduction of all terms with factors h^n up to h^3 inclusive. The order of the system of equations (11.21a) - (11.21b) is lower than the order of the initial system (11.1a) - (11.1b). The order of the initial system, as is obvious, is 21. Here the higher derivatives in eqs.(11.1a) and (11.1b) will be mixed derivatives with respect to t and the coordinates x^i (i = 1, 2). The system of equations (11.1a) - (11.1b) will be of the fifteenth order in the derivatives with respect to the coordinates x^i. The lowering of the order of the system as a result of the application of the method of successive approximations is due to elimination of the terms containing factors of h^n where $n \geq 4$, in introducing the formulas of reduction in eqs.(11.1a) - (11.1b).

2. The system of equations (11.21a) - (11.21b) approximately describes the dynamic process in elastic shells of arbitrary form. As we have already noted, these equations contain all terms with factors h^n where $n \leq 3$. Neglecting the remaining terms naturally limits the significance of the derived equations. This fact will become obvious when considering the boundary and initial conditions.

3. The system of equations (11.21a) - (11.21b) "symbolically" is resolved into a system of two equations containing the tangential components of the displacement vector and an equation with the normal component. This resolution, however, is illusory since the covariant derivatives, forming the basis of the operators introduced by us to shorten the notation, are sets containing all com-

ponents of the vector $\vec{u}$. Only in the plane problem is this resolution actually accomplished.

4. The system of equations (11.21a) - (11.21b) contains the wave operators M. However, the question of the existence of the actual characteristics of eqs.(11.21a) - (11.21b) requires separate analysis.

5. The system of equations (11.21a) - (11.21b) obtained by our analytic methods is very complex and permits only approximate integration, neglecting a number of terms. It seems useful, however, to introduce such a system into the arsenal of descriptive means of the theory of shells as a peculiar "stage", /113
permitting us to judge the accuracy of the equations obtained by other, more pictorial methods*.

Equations (11.21a) - (11.21b) do not define the statement of the dynamic boundary conditions of the shell theory. The boundary and initial conditions must be considered. In order to do this, we must first find approximation expressions for the stress tensor components.

Section 12. Approximate Expressions for the Components of the Displacement Vector and the Components of the Stress Tensor

In considering the expansions of the displacement vector components, the question arises as to the number of terms that must be retained in these expansions.

Based on the relative accuracy of eqs.(11.21a) - (11.21b), we will retain in the expansions of the displacement vector components all terms including components with factors z^3. Here, however, we have a certain inconsistency, since eqs.(11.21a) - (11.21b) contain terms depending on coefficients of z^4 in the expansions of the displacement vector components. This inconsistency, however, is apparently one of several contradictions of the theory under consideration. Below, we will discuss the contradictions in the approximate theory of shells in more detail. In the notation adopted by us we find

$$u_i^{(z)} = u_i + z\,[u_i^{(1)}]_2 + \frac{1}{2} z^2 [u_i^{(2)}]_1 + \frac{1}{6} z^3 [u_i^{(3)}]_1 + \dots \quad (i = 1, 2, 3). \tag{12.1}$$

Making use of the relations (11.8a) - (11.9b), we obtain

$$u_i^{(z)} = u_i + z\,\{[S_i^{(1)}]_2 u_3 + [Q_i^{(1)}]_2\} + \frac{1}{2} z^2 \{[P^{(2)}]_1 u_i +$$

* The desirability of investigations to obtain arbitrarily "exact" equations of the shell theory, permitting a judgment from the properties of the rejected terms, was discussed at the Conference on Shell Theory held in October 1960 at Kazan'.

$$+[R_i^{(2)s}]_1 u_s+[Q^{(2)}]_1\}+\frac{1}{6}z^3\{[S_i^{(3)}]_1 u_3+[Q^{(3)}]_1\}+\ldots; \quad (12.2a)$$

$$u_3^{(z)}=u_3+z\{[R_3^{(1)s}]_2 u_s+[Q_3^{(1)}]_2\}+\frac{1}{2}z^2\{[S_3^{(2)}]_1 u_3+$$
$$+[Q^{(2)}]_1\}+\frac{1}{6}z^3\{[R_3^{(3)s}]_1 u_s+[Q_3^{(3)}]_1\}+\ldots$$
$$(i, s=1, 2). \quad (12.2b)$$

Consider the expansions of the stress tensor components. Introducing /114
the notation of eq.(11.5), we find

$$\sigma_{ik}^{(z)}=\lambda g_{ik}[u_3^{(1)}]_2-\frac{\lambda(\lambda+2\mu)}{\lambda+\mu}g_{ik}N_{3.}^{\cdot s}u_s-\frac{\mu^2}{\lambda+\mu}(L_i u_k+L_k u_i)+$$
$$+z\left\{\lambda g_{ik}[u_3^{(2)}]_2-\frac{\lambda(\lambda+2\mu)}{\lambda+\mu}g_{ik}N_{3.}^{\cdot s}[u_s^{(1)}]_2-\frac{\mu^2}{\lambda+\mu}(L_i[u_k^{(1)}]_2+\right.$$
$$\left.+L_k[u_i^{(1)}]_2)\right\}+\frac{1}{2}z^2\left\{\lambda g_{ik}[u_3^{(3)}]_1-\frac{\lambda(\lambda+2\mu)}{\lambda+\mu}g_{ik}N_{3.}^{\cdot s}[u_s^{(2)}]_1-\right.$$
$$\left.-\frac{\mu^2}{\lambda+\mu}(L_i[u_k^{(2)}]_1+L_k[u_i^{(2)}]_1)\right\}+\frac{1}{6}z^3\left\{\lambda g_{ik}[u_3^{(4)}]_1-\right.$$
$$\left.-\frac{\lambda(\lambda+2\mu)}{\lambda+\mu}g_{ik}N_{3.}^{\cdot s}[u_s^{(3)}]_1-\frac{\mu^2}{\lambda+\mu}(L_i[u_k^{(3)}]_1+L_k[u_i^{(3)}]_1)\right\}+\ldots \quad (12.3a)$$
$$(i, k, s=1, 2).$$

$$\sigma_{i3}^{(z)}=\mu\left\{[u_i^{(1)}]_2-\frac{\mu}{\lambda+\mu}L_i u_3\right\}+\mu z\left\{[u_i^{(2)}]_2-\frac{\mu}{\lambda+\mu}L_i[u_3^{(1)}]_2\right\}+$$
$$+\frac{1}{2}\mu z^2\left\{[u_i^{(3)}]_1-\frac{\mu}{\lambda+\mu}L_i[u_3^{(2)}]_1\right\}+\frac{1}{6}\mu z^3\left\{[u_i^{(4)}]_1-\right.$$
$$\left.-\frac{\mu}{\lambda+\mu}L_i[u_3^{(3)}]_1\right\}+\ldots, \quad (12.3b)$$

$$\sigma_{33}^{(z)}=(\lambda+2\mu)\left\{[u_3^{(1)}]_2-\frac{\lambda}{\lambda+\mu}N_{3.}^{\cdot s}u_s\right\}+(\lambda+2\mu)z\left\{[u_3^{(2)}]_2-\right.$$
$$\left.-\frac{\lambda}{\lambda+\mu}N_{3.}^{\cdot s}[u_s^{(1)}]_2\right\}+\frac{1}{2}(\lambda+2\mu)z^2\left\{[u_3^{(3)}]_1-\frac{\lambda}{\lambda+\mu}N_{3.}^{\cdot s}[u_s^{(2)}]_1\right\}+$$
$$+\frac{1}{6}(\lambda+2\mu)z^3\left\{[u_3^{(4)}]_1-\frac{\lambda}{\lambda+\mu}N_{3.}^{\cdot s}[u_s^{(3)}]_1\right\}+\ldots$$
$$(i, s=1, 2). \quad (12.3c)$$

Equations (11.8a) - (11.9b), together with the values of their operators expressed by eqs.(11.10a) - (11.18b), permit the approximate representation of the stress tensor components in a form analogous to eqs.(12.2a) - (12.2b). We shall not write out these expressions in view of their great length. With respect to the expressions found by us for the components of the displacement vector and of the stress tensor, we may remark that they contain terms depending on the acceleration of an element of the shell. These terms will hereafter be designated "inertial".

The presence of inertial terms distinguishes our approximation expressions for the displacement vector components and the stress tensor from the expressions known from the classical theory. It is obvious that these expressions /115 contain a number of non-inertial terms, which are also absent from the relations of the classical theory.

Section 13. Boundary Conditions

The equations of motion of an element of the shell were obtained by us from the equations of motion of a three-dimensional body. It was natural at first sight to turn to the boundary conditions of the three-dimensional problem of the theory of elasticity to obtain the boundary conditions of the theory of shells. This is exactly what we did. Consider two fundamental boundary problems. In the first problem the displacements on the contour surface are prescribed and, in the second problem, the stresses (II, Sect.8). The contour surface will be analytically determined by the following conditions imposed on the unit vector $\vec{n}$ of the external normal:

$$n_3 = n^3 = 0. \tag{13.1}$$

1. The First Boundary Condition

On the contour surface C, let the displacements

$$(u_i^{(z)})_C = \varphi_i(x^j, z, t) \qquad (i = 1, 2, 3; \quad j = 1, 2). \tag{13.2}$$

be prescribed. Expanding the prescribed displacements in tensor series in powers of z, we find

$$(u_i^{(z)})_C = \varphi_i(x^j, 0, t) + z\varphi_i^{(1)}(x^j, 0, t) + \frac{1}{2}z^2\varphi_i^{(2)}(x^j, 0, t) + \\ + \frac{1}{6}z^3\varphi_i^{(3)}(x^j, 0, t) + \dots \tag{13.3}$$

Equating the first four terms of the expansion (13.3) to the first four terms of the expansions (12.2a) - (12.3b), we find

$$(u_i)_C = \varphi_i(x^j, 0, t); \quad ([S_i^{(1)}]_2 u_3 + [Q_i^{(1)}]_2)_C = \varphi_i^{(1)}(x^j, 0, t),$$
$$([P^{(2)}]_1 u_i + [R_i^{(2)s}]_1 u_s + [Q^{(2)}]_1)_C = \varphi_i^{(2)}(x^j, 0, t), \tag{13.4a}$$
$$([S_i^{(3)}]_1 u_3 + [Q^{(3)}]_1)_C = \varphi_i^{(3)}(x^j, 0, t);$$

$$(u_3)_C = \varphi_3(x^j, 0, t); \quad ([R_3^{(1)s}]_2 u_s + [Q_3^{(1)}]_2)_C = \varphi_3^{(1)}(x^j, 0, t),$$
$$([S_3^{(2)}]_1 u_3 + [Q^{(2)}]_1)_C = \varphi_3^{(2)}(x^j, 0, t),$$
$$([R_3^{(3)s}]_1 u_s + [Q_3^{(3)}]_1)_C = \varphi_3^{(3)}(x^j, 0, t) \tag{13.4b}$$
$$(i, j, s = 1, 2).$$

where C is an arc of the contour of the basic surface of the shell. The conditions (13.4a) - (13.4b) were obtained by us as a result of a formal operation. The total number of these conditions is twelve.

We have two remarks to make on the conditions (13.4a) - (13.4b). /116

1. The compatibility of the conditions (13.4a) - (13.4b) with eqs.(11.21a) - (11.22b) is not obvious. Apparently some of these conditions (13.4a) - (13.4b) cannot be satisfied by solutions of the system of equations (11.21a) - (11.21b). In fact, the order of the system of equations (11.21a) - (11.21b) is twelve. If we recall that the solution of a partial differential equation of second order permits satisfaction of one boundary condition*, while the solution of a biharmonic equation satisfies two boundary conditions, then the solutions of the system of equations (11.21a) - (11.21b) must satisfy six boundary conditions. In other words, we shall have to confine ourselves to two terms in the expansions (13.3) and accordingly to two terms in the expansions (12.2a) - (12.2b).

Obviously, the arguments presented here are not rigorous. The mentioned questions require special investigation. We shall return to them later.

2. In problems of the shell theory, the functions $\varphi(x^j, z, t)$ are usually not prescribed but it is required to satisfy, by conditions imposed on the wanted functions, weaker restrictions on the contour of the basic surface. Thus, the above-mentioned difficulties do not arise in practice.

2. Second Boundary Problem

Let us assume that, on the contour surface, the stress vector

$$f_i = f_i(x^j, z, t) \quad (i = 1, 2, 3). \tag{13.5}$$

is prescribed. Expanding this vector in a tensor Taylor** series in powers of z,

* The Dirichlet and Neumann problems are examples.

** We recall that an expansion in a tensor power series brings about the operation of parallel displacement in the Levi-Civita sense.

we find

$$f_i(x^j, z, t) = f_i(x^j, 0, t) + z f_i^{(1)}(x^j, 0, t) + \frac{1}{2} z^2 f_i^{(2)}(x^j, 0, t) + \\ + \frac{1}{6} z^3 f_i^{(3)}(x^j, 0, t) + \dots \quad \text{(a)}$$

Let us now make use of eq.(II, 8.2b):

$$\sigma_{ik} n^k = f_i \qquad (i, k = 1, 2, 3). \quad \text{(b)}$$

where σ_{ik} and n^k are, respectively, the components of the stress tensor and of the unit vector $\vec{n}$ of the exterior normal to the contour surface, displaced parallel to themselves on the basic surface of the shell along the normal to this surface. /117

The parallel displacement of the stress tensor is accomplished by expanding its components in tensor series. The parallel displacement of the vector $\vec{n}$ is performed on the basis of previous statements (I, Sect.11). The possibility of a separate displacement of the stress tensor $^2\vec{\sigma}$ and the vector $\vec{n}$ results from the fundamental properties of the operation of parallel displacement in the Levi-Civita sense (I, Sect.10).

It follows from (I, 11.13) that the relations (13.1) remain valid for the displaced vector $\vec{n}$. The remaining components of the displaced vector $\vec{n}$ are determined by equations resulting from (I, 11.13) and (I, 11.18):

$$n^i = n^i_{(z)}(1 - k_i z); \quad n^3 = n_3 = 0 \quad (13.6)$$

(i = 1, 2; do not sum over i!).
Here, $n^i_{(z)}$ are the components of the unit vector $\vec{n}$ to the contour surface at the point of the shell with the coordinate $x^3 = z$.

Let us expand $n^i_{(z)}$ in a Taylor series in powers of z. Then eqs.(13.6) lead to the following expressions:

$$n^i = n^i_{(0)} + z n^i_{(1)} + z^2 n^i_{(2)} + z^3 n^i_{(3)} + \dots \quad (13.7)$$

To set up the boundary conditions we must bear in mind eqs.(a) and the expansions (12.3a) - (12.3c), eq.(13.5), and formula (13.7). We have

$$\sigma_{11} n^1 + \sigma_{12} n^2 = f_1; \quad \sigma_{21} n^1 + \sigma_{22} n^2 = f_2; \quad \sigma_{31} n^1 + \sigma_{32} n^2 = f_3. \quad (13.8)$$

In the expanded form, these equations after equating the coefficients of equal powers of z on the left and right-hand sides lead to the following system of boundary conditions:

$$\sum_{k=0}^{p}\left\{\lambda g_{ir}\left[u_3^{(k+1)}\right]_m-\frac{\lambda(\lambda+2\mu)}{\lambda+\mu}g_{ir}N_{3.}^{.s}\left[u_s^{(k)}\right]_m-\right.$$

$$\left.-\frac{\mu^2}{\lambda+\mu}\left(L_i\left[u_r^{(k)}\right]_m+L_r\left[u_i^{(k)}\right]_m\right)\right\}n^r_{(p-k)}=f_i^{(p)}(x^j,0,t), \tag{13.9a}$$

$$\mu\sum_{k=0}^{p}\left\{\left[u_r^{(k+1)}\right]_m-\frac{\mu}{\lambda+\mu}L_r\left[u_3^{(k)}\right]_m\right\}n^r_{(p-k)}=f_3^{(p)}(x^j,0,t) \tag{13.9b}$$

$$(p=0,1,2,3;\quad i,j,r,s=1,2).$$

The selection of the approximation m is so performed that, in the conditions (13.8), no terms of the "order" h^4 will enter. The orders of h and z are taken to be the same*. We assume that $u_q^{(0)} = u_q$. The summation over r is denoted by the usual convention. /118

The system of relations (13.9a) - (13.9b) contains twelve conditions. The above remark 1, on the number of conditions of the boundary problems of shell theory in our formulation, also applies here.

The question as to the number and meaning of the boundary conditions in shell theory is not new. Over a hundred years ago there was a discussion between the followers of Poisson's theory, according to whom five force conditions had to be satisfied on the contour of the central plane or middle surface of a plate, and adherents of the theory of Kirchhoff, who asserted that the number of these conditions did not exceed four. The Kirchhoff theory, using the well-known simplifying static-geometrical hypotheses, is generally recognized at the present time. The impossibility of satisfying all the boundary conditions of the first or second boundary problems of shell theory naturally leads to the idea that there must be some internal contradiction in the theory developed by us as a whole.

Indeed, the accuracy of the boundary condition that can be satisfied will be lower than the accuracy of the system of equations (11.21a) - (11.21b), which naturally raises the question whether these equations are not excessively accurate and unjustifiably complex.

It is, however, easy to prove that the theory developed here contains no logical contradictions. We shall return later to its evaluation. However, these and similar questions encourage the study of other analytical approaches to the mathematical description of the stressed and strained state of shells. One of them is based on the use of the variational principles of the mechanics of elastic bodies**. We shall make use of this method later, and shall then

* In other words, terms with factors $h^m z^n$, where $m + n > 3$, must not enter into the equations.

** V.V.Bolotin has called our attention to the advantage of making extensive use of variational methods in the general theory of shells.

return to the general analysis of the formulation of the boundary problems of the dynamics of shells.

Section 14. Initial Conditions. General Remarks on the First Version of the Solution of the Problem of Reduction

To complete our brief outline of the general formulation of the dynamic boundary problems of the theory of shells in the first version, let us consider the initial conditions. We shall start out from the initial conditions of the dynamics of a three-dimensional elastic body (II, 8.1a-b). Let

$$u_{i0}(x^j) = \psi_i(x^j, z); \quad u_{i0} = \theta_i(x^j, z). \tag{14.1}$$

Expanding these vectors in tensor series in powers oı z, we find /119

$$\psi_i(x^j, z) = \psi_i(x^j, 0) + z\psi_i^{(1)}(x^j, 0) + \frac{1}{2} z^2\psi_i^{(2)}(x^j, 0) + \\ + \frac{1}{6} z^3\psi_i^{(3)}(x^j, 0) + \dots, \tag{14.2a}$$

$$\theta_i(x^j, z) = \theta_i(x^j, 0) + z\theta_i^{(1)}(x^j, 0) + \frac{1}{2} z^2\theta_i^{(2)}(x^j, 0) + \\ \frac{1}{6} z^3\theta_i^{(3)}(x^j, 0) + \dots \qquad (i = 1, 2, 3). \tag{14.2b}$$

Making use of the expansions (12.2a) - (12.2b), we find the following initial conditions:

$$\begin{gathered} u_i |_{t=t_0} = \psi_i(x^j, 0); \quad [S_i^{(1)}]_2 u_3 + [Q_i^{(1)}]_2 |_{t=t_0} = \psi_i^{(1)}(x^j, 0); \\ [P^{(2)}]_1 u_i + [R_i^{(2)s}]_1 u_s + [Q^{(2)}]_1 |_{t=t_0} = \psi_i^{(2)}(x^j, 0); \\ [S_i^{(3)}]_1 u_3 + [Q^{(3)}]_1 |_{t=t_0} = \psi_i^{(3)}(x^j, 0). \end{gathered} \tag{14.3a}$$

$$\begin{gathered} u_i |_{t=t_0} = \theta_i(x^j, 0); \quad [\dot S_i^{(1)}]_2 u_3 + [\dot Q_i^{(1)}]_2 |_{t=t_0} = \theta_i^{(1)}(x^j, 0); \\ [\dot P^{(2)}]_1 u_i + [\dot R_i^{(2)s}]_1 u_s + [\dot Q^{(2)}]_1 |_{t=t_0} = \theta_i^{(2)}(x^j, 0); \\ [\dot S_i^{(3)}]_1 u_3 + [\dot Q^{(3)}]_1 |_{t=t_0} = \theta_i^{(3)}(x^j, 0). \end{gathered} \tag{14.3b}$$

$$\begin{gathered} u_3 |_{t=t_0} = \psi_3(x^j, 0); \quad [R_3^{(1)s}]_2 u_s + [Q_3^{(1)}]_2 |_{t=t_0} = \psi_3^{(1)}(x^j, 0); \\ [S_3^{(2)}]_1 u_3 + [Q^{(2)}]_1 |_{t=t_0} = \psi_3^{(2)}(x^j, 0); \\ [R_3^{(3)s}]_1 u_s + [Q_3^{(3)}]_1 |_{t=t_0} = \psi_3^{(3)}(x^j, 0). \end{gathered} \tag{14.3c}$$

$$
\begin{aligned}
&\dot{u}_3\big|_{t=t_0}=\theta_3(x^j,0);\quad [\dot{R}_3^{(1)s}]_2 u_s+[\dot{Q}_3^{(1)}]_2\big|_{t=t_0}=\theta_3^{(1)}(x^j,0);\\
&[\dot{S}_3^{(2)}]_1 u_3+[\dot{Q}^{(2)}]_1\big|_{t=t_0}=\theta_3^{(2)}(x^j,0);\\
&[\dot{R}_3^{(3)s}]_1 u_s+[\dot{Q}_3^{(3)}]\big|_{t=t_0}=\theta_3^{(3)}(x^j,0)\\
&\qquad\qquad (i,j,s=1,2).
\end{aligned}
\tag{14.3d}
$$

In all, twenty-four initial conditions (14.3a) - (14.3d) must be satisfied. From the expressions for the operators (11.10a) - (11.18b), it is clear that the system of equations (11.21a) - (11.21b) is of the order twelve with respect to the time t. Each of the equations (11.21a) - (11.21b) is of the fourth order in t, containing the wave operator M^2. It is clear from this that the solutions of the system of equations (11.21a) - (11.21b) can satisfy only twelve initial conditions. The remaining twelve conditions will not be satisfied. Consequently, the solutions of the system (11.21a) - (11.21b) cannot, with the accuracy prescribed by us, i.e., with an accuracy to terms of the "order" h^3, describe the initial distribution of the displacements and velocities in the shell*. Obviously, even in future motion, the solutions of the system of equations (11.21a) - (11.21b) will not describe the fields of displacements and velocities with the required accuracy. /120

All this forces us to conclude that satisfaction of the boundary and initial conditions with an accuracy to terms of the "order" h^3 is possible only if the order of this system of equations (11.21a) - (11.21b) is increased, which can be accomplished by introducing into these equations terms with factors h^4, h^5, h^6, and h^7. The system of equations (11.21a) - (11.21b) with terms to the "order" h^3 inclusive may be useful in the study of dynamic processes that do not require rigorous satisfaction of the initial and boundary conditions. These problems include the problem of the propagation of perturbations in unbounded rods, plates and shells, the problem of local and very brief influences caused by impact, etc.

This method permits obtaining eqs.(11.21a) - (11.21b) that contain terms which can be interpreted to be a result of the influence of shear stresses $\sigma_{i3}^{(z)}$ and of the inertia of rotation of an element of the shell**. The appearance of these terms in eqs.(11.21a) - (11.21b) involves none of the kinetic-geometrical hypotheses that have been introduced in a number of modern works, but is instead the result of the analytic construction of eqs.(11.21a) - (11.21b).

To summarize, it may be said that the above method of expansion in series corresponding to the best approximation of the required functions "at a point" permits us to construct*** a logically non-contradictory technique for reducing the three-dimensional problems of the theory of elasticity to two-dimensional

* In the absence of surface forces, eqs.(11.21a) - (11.21b) will contain terms with the factors h^0 and h^2

** See, for instance, the above-cited dissertations by M.P.Petrenko and I.T.Selezov.

*** The optimum representation of "in-the-mean" functions and its application to shell theory will be discussed later.

problems.

We note two shortcomings of the method.

1. The satisfaction of boundary and initial conditions with a prescribed accuracy by convention requires a relatively high accuracy of eqs.(11.21a) - (11.21b). For example, to satisfy the boundary and initial conditions with an accuracy to terms containing factors of the order of z^3 requires us to retain terms with factors up to h^7 inclusive, in eqs.(11.21a) - (11.21b). This shortcoming is in part due to the iteration process employed by us, which lowers the order of the system of equations (7.4a) - (7.4d). However, as will be clear from the concluding remarks to Section 11, the order of the system of equation (7.4a) - (7.4d) is also insufficient to satisfy the boundary and initial con- /121 ditions with an arbitrary accuracy equal to the arbitrary accuracy of these equations.

Consequently, relatively slight errors in the preliminary determination of the stress tensor components σ_{13} lead to greater errors in the subsequent determination of the fields of displacement, the velocity of displacement, and the stress tensor as a whole.

The index of variability is of considerable significance in the problem of setting up approximation formulas that describe kinetic phenomena in shells with sufficient accuracy (Bibl.5, 27, 29). According to another author(Bibl.27) we may assume that neglecting the terms that contain the factor h^n will introduce an error of the order of $\frac{r^n h^n}{a^n}$, where r is the index of variability, and a is a dimension characteristic for the basic surface of the shell. But the question of evaluating the error may become more complicated when we consider the solutions of refined equations. This is confirmed by the existence of boundary effects of new types, discovered on an analysis of the solution of the refined statical equations given by E.Reissner (Bibl.20a).

Questions connected with the characterization of the accuracy of approximate dynamic equations by means of the index of variability are still in the stage of study, and we shall not consider them here*.

2. The system of equations (11.21a) - (11.21b) is very complicated. It is entirely possible that there exist simplifications of this system, which have only a negligible effect on the fields of displacement and stress. The method employed gives no answer to this question.

Let us pass now to other methods of solving the reduction problem and of formulating the dynamic boundary problems of shell theory.

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* The status of the problem at the present time is given by another author (Bibl.20b). The complexity of the problem is increased by the introduction of inertial terms into the boundary conditions, when certain methods of reduction are used.

Section 15. Application of the General Equation of Dynamics to the Solution of the Problem of Reduction

Let us make use of the general equation of dynamics, set up with respect to the motion of an elastic body*:

$$\iiint\limits_{(V)} \rho\left(F_i - \frac{\partial^2 u_i}{\partial t^2}\right)\delta u^i dV + \iint\limits_{(S)} X_i \delta u^i dS - \delta A = 0. \tag{15.1}$$

where X_i are the forces acting on the surface S on the body, and δA is the elementary work of deformation defined by (II, 11.1). The other notations are familiar.

We recall that eq.(15.1) includes all the forms of the equation of motion of an elastic body. Equation (15.1) yields the solution of the reduction problem and makes it possible to formulate the dynamic boundary problems of the theory of shells. The fundamental method of reduction resulting from eq.(15.1) is an approximation of the components of the displacement vector and the stress tensor by finite sums of functions of the coordinate z, selected in a definite way and having coefficients depending on the interior coordinates x^j of the basic surface of the shell. /122

This scheme includes most of the methods known today for solving the problem of reduction of a three-dimensional problem of the theory of elasticity to a two-dimensional problem. An exception is the method considered in the last few Sections, since it does not involve an integration of approximation functions over the coordinate z.

The possibility of applying the general equation of dynamics to the solution of the problem of reducing the three-dimensional static problem of the theory of elasticity to a two-dimensional problem of the theory of shells has been noted by V.Z.Vlasov in his monograph (Bibl.3a). Kh.M.Mushtari and I.G.Teregulov discuss this problem in the nonlinear formulation in great detail (Bibl.27).

The method of reduction indicated in a monograph (Bibl.3a) differs from the method used in another paper (Bibl.27) as well as from the method developed by us below, in that it is less general. We compare these methods in more detail in Section 30.

Consider in succession the quantities entering into eq.(15.1). The element of volume dV and the element of area $dS_{(\pm)}$ of the boundary surfaces of the shell are expressed by the following equations:

$$dV = \sqrt{g}\, dx^1 dx^2 dz = \sqrt{g_{11} g_{22}}\,(1 - k_1 z)(1 - k_2 z)\, dx^1 dx^2 dz, \tag{15.2}$$

---

* Cf., for example, L.S.Leybenzon, Collection of Works, Vol.1, pp.193-194. USSR Academy of Sciences, 1951.

$$dS_{(\pm)} = \sqrt{g_{11}g_{22}}\,(1 \mp k_1 h)(1 \mp k_2 h)\,dx^1 dx^2. \qquad (15.3)$$

We confine ourselves here to the consideration of shells of constant thickness 2h. The relations (I, 2.6b) and (I, 3.6a - 3.6b) are used here (Bibl.13). The element of area dS_C of the contour surface C is defined by the equation

$$dS_C = \sqrt{g_{11}(1-k_1 z)^{(2)}(\dot{x}^1)^2 + g_{22}(1-k_2 z)^{(2)}(\dot{x}^2)^{(2)}}\,du\,dz, \qquad (15.4)$$

where

$$x^i = x^i(u) \quad (i = 1, 2) \qquad (a)$$

are the parametric equations of the contour of the basic surface of the shell. Since we will make use of segments of tensor power series, which approximately determine the vector u_i, the variations δu^i, and the stress tensor σ^{ik} on the basic surface, we shall displace the vectors ρF_i and X_i to the basic surface, using the operators of parallel displacement (I, 11.20). To avoid complicating the formulas, we shall retain the previous notation for the displaced ρF_i and /123 X_i. We put further

$$u_i^{(z)} = u_i + z u_i^{(1)} + \frac{1}{2} z^2 u_i^{(2)} + \frac{1}{3!} z^3 u_i^{(3)} + \ldots . \qquad (15.5)$$

The quantities $u_i^{(1)}$, $u_i^{(2)}$, $u_i^{(3)}$... are the generalized coordinates of the shell.

We shall hereafter confine ourselves to the same conditional accuracy adopted by us in considering the first version of the solution of the reduction problem. In view of the fact that eq.(15.5) determines the vector $u_i^{(z)}$, displaced to the basic surface, we find

$$\delta u^i = g_{(z)}^{ik}\,\delta u_k^{(z)} = g^{ii}\left[\delta u_i + z\delta u_i^{(1)} + \frac{1}{2} z^2 \delta u_i^{(2)} + \frac{1}{3!} z^3 \delta u_i^{(3)} + \ldots\right] \qquad (15.6)$$

(do not sum over i !).

Consider now δA. By (II, 11.1) we have*

$$\delta A = \iiint\limits_{(V)} \sigma^{ik}\delta D_{ik}\,dV. \qquad (15.7)$$

Let us now take up the transformation of the sum

$$\delta W^{(z)} = \sigma_{(z)}^{ik}\,\delta D_{ik}^{(z)}. \qquad (15.8)$$

* Cf. also the above-cited work by L.S.Leybenzon.

Bearing in mind the commutativity of the operations of variations and covariant differentiation,

$$\delta\nabla_i u_k^{(n)} = \nabla_i \delta u_k^{(n)}, \tag{15.9}$$

we obtain

$$\delta W^{(z)} = \frac{1}{2}\,\sigma_{(z)}^{ik}\left(\nabla_i \delta u_k^{(z)} + \nabla_k \delta u_i^{(z)}\right) = \sigma_{(z)}^{ik}\,\nabla_i \delta u_k^{(z)}. \tag{15.10}$$

We find, further,

$$\begin{aligned}
\delta W^{(z)} &= \delta W + z\delta W^{(1)} + \frac{1}{2} z^2 \delta W^{(2)} + \frac{1}{3!} z^3 \delta W^{(3)} + \ldots = \\
&= \sigma^{ik}\nabla_i \delta u_k + \sigma^{i3}\left(\nabla_i \delta u_3 + \delta u_i^{(1)}\right) + \sigma^{33}\delta u_3^{(1)} + \\
&+ z\left[\sigma_{(1)}^{ik}\,\nabla_i \delta u_k + \sigma_{(1)}^{i3}\left(\nabla_i \delta u_3 + \delta u_i^{(1)}\right) + \sigma_{(1)}^{33}\delta u_3^{(1)} + \right.\\
&\left. + \sigma^{ik}\nabla_i \delta u_k^{(1)} + \sigma^{i3}\left(\nabla_i \delta u_3^{(1)} + \delta u_i^{(2)}\right) + \sigma^{33}\delta u_3^{(2)}\right] + \\
&+ \frac{1}{2} z^2\left[\sigma_{(2)}^{ik}\nabla_i \delta u_k + \sigma_{(2)}^{i3}\left(\nabla_i \delta u_3 + \delta u_i^{(1)}\right) + \sigma_{(2)}^{33}\delta u_3^{(1)} + \right.\\
&+ 2\sigma_{(1)}^{ik}\nabla_i \delta u_k^{(1)} + 2\sigma_{(1)}^{i3}\left(\nabla_i \delta u_3^{(1)} + \delta u_i^{(2)}\right) + 2\sigma_{(1)}^{33}\delta u_3^{(2)} + \\
&\left. + \sigma^{ik}\nabla_i \delta u_k^{(2)} + \sigma^{i3}\left(\nabla_i \delta u_3^{(2)} + \delta u_i^{(3)}\right) + \sigma^{33}\delta u_3^{(3)}\right] + \\
&+ \frac{1}{3!} z^3\left[\sigma_{(3)}^{ik}\,\nabla_i \delta u_k + \sigma_{(3)}^{i3}\left(\nabla_i \delta u_3 + \delta u_i^{(1)}\right) + \sigma_{(3)}^{33}\,\delta u_3^{(1)} + \right.\\
&+ 3\sigma_{(2)}^{ik}\nabla_i \delta u_k^{(1)} + 3\sigma_{(2)}^{i3}\left(\nabla_i \delta u_3^{(1)} + \delta u_i^{(2)}\right) + 3\sigma_{(2)}^{33}\delta u_3^{(2)} + \\
&+ 3\sigma_{(1)}^{ik}\Delta_i \delta u_k^{(2)} + 3\sigma_{(1)}^{i3}\left(\nabla_i \delta u_3^{(2)} + \delta u_i^{(3)}\right) + 3\sigma_{(1)}^{33}\delta u_3^{(3)} + \\
&\left. + \sigma^{ik}\nabla_i \delta u_k^{(3)} + \sigma^{i3}\left(\nabla_i \delta u_3^{(3)} + \delta u_i^{(4)}\right) + \sigma^{33}\delta u_3^{(4)}\right] + \ldots \\
&\qquad (i,\, k = 1,\, 2).
\end{aligned} \tag{15.11}$$

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The coefficients $\sigma_{(n)}^{ik}$ of the expansions of the stress tensor in powers of z have the following meaning:

$$\sigma_{(n)}^{ik} = \lambda g^{ik} g^{rr}\nabla_r u_r^{(n)} + \lambda g^{ik} u_3^{(n+1)} + \mu g^{ii} g^{kk}\left(\nabla_i u_k^{(n)} + \nabla_k u_i^{(n)}\right); \tag{15.12a}$$

$$\sigma_{(n)}^{i3} = \mu g^{ii}\left(\nabla_i u_3^{(n)} + u_i^{(n+1)}\right); \tag{15.12b}$$

$$\sigma_{(n)}^{33} = \lambda g^{rr}\nabla_r u_r^{(n)} + (\lambda + 2\mu)\, u_3^{(n+1)} \tag{15.12c}$$

$(n = 0, 1, 2, 3\cdot \quad i, k, r = 1, 2;$ do not sum over i and k!).

Here,

$$\sigma_{(0)}^{\alpha\beta} = \sigma^{\alpha\beta} \quad (\alpha, \beta = 1, 2, 3) \tag{15.13}$$

are the components of the stress tensor on the basic surface.

To prepare all the summands entering into the variational equation (15.1) for the forthcoming transformation, let us consider the sum $\frac{\partial^2 u_i}{\partial t^2}\,\delta u^i$, retaining in it all terms up to terms with the factor z^3 inclusive. Making use of eqs.(15.5) - (15.6), we find

$$\frac{\partial^2 u_i}{\partial t^2}\delta u^i = g^{ii}\Big[\ddot{u}_i\delta u_i + z(\ddot{u}_i^{(1)}\delta u_i + \ddot{u}_i\delta u_i^{(1)}) + \frac{1}{2}z^2(\ddot{u}_i^{(2)}\delta u_i + 2\ddot{u}_i^{(1)}\delta u_i^{(1)} + \ddot{u}_i\delta u_i^{(2)}) + \frac{1}{3!}z^3(\ddot{u}_i^{(3)}\delta u_i + 3\ddot{u}_i^{(1)}\delta u_i^{(2)} + 3\ddot{u}_i^{(2)}\delta u_i^{(1)} + \ddot{u}_i\delta u_i^{(3)}) + \ldots\Big] \quad (i = 1, 2, 3). \tag{15.14}$$

Differentiation with respect to time is denoted here and hereafter by tremas.

Now let us substitute the expressions (15.3), (15.4), (15.6), (15.11), and (15.14) into the variational equation (15.1). Let us integrate over z, under the assumption that the basic surface coincides with the central plane of the shell, and confining ourselves to summands with the factors h, h^2, and h^3. We introduce the notation /125

$$Q^{(m)\,i} = \frac{1}{m!g_{ii}}\Big[\int_{-h}^{+h}\rho F_i z^m (1-k_1 z)(1-k_2 z)\,dz + X_{(+)\,i}(1-k_1 h)(1-k_2 h)h^m - (-1)^m X_{(-)\,i}(1+k_1 h)(1+k_2 h)h^m\Big]; \tag{15.15a}$$

$$S^{(m)\,i} = \frac{1}{m!g^{ii}}\int_{-h}^{+h} z^m X_i \sqrt{\frac{g_{11}(1-k_1 z)^{(2)}(\dot{x}^1)^{(2)} + g_{22}(1-k_2 z)^{(2)}(\dot{x}^2)^{(2)}}{g_{11}(\dot{x}^1)^{(2)} + g_{22}(\dot{x}^2)^{(2)}}}\,dz \quad (i = 1, 2, 3). \tag{15.15b}$$

Here we made use of eq.(15.4). The components of the stress vector X_i can be expressed, if convenient, in terms of the stress tensor by eqs.(13.8).

We recall again that when we apply the general equation of dynamics, all vectors of forces are first displaced to the basic surface by means of the operators of parallel displacement (I, 11.22). The quantities $Q^{(m)\,i}$ and $S^{(m)\,i}$ are the respective generalized forces on the basic and contour surfaces corre-

sponding to the generalized coordinates $u_i^{(n)}$.

After several transformations and application of the Ostrogradskiy-Gauss theorem, the general equation of dynamics (15.1) takes the following approximate form*:

$$
\begin{aligned}
&\iint\limits_{(\omega)} \bigg\{ \bigg[Q^{(0)k} + \nabla_i \left(2h + \frac{2}{3} k_1 k_2 h^3\right) \sigma^{ik} - \frac{2}{3} h^3 \nabla_i (k_1 + k_2) \sigma^{ik}_{(1)} + \\
&+ \frac{1}{3} h^3 \nabla_i \sigma^{ik}_{(2)} - \rho g^{kk} \left(2h + \frac{2}{3} k_1 k_2 h^3\right) \ddot{u}_k + \frac{2}{3} \rho h^3 (k_1 + k_2) g^{kk} \ddot{u}_k^{(1)} - \\
&- \frac{1}{3} h^3 \rho g^{kk} \ddot{u}_k^{(2)} \bigg] \delta u_k + \bigg[Q^{(0)3} + \nabla_i \left(2h + \frac{2}{3} k_1 k_2 h^3\right) \sigma^{i3} - \\
&- \frac{2h^3}{3} \nabla_i (k_1 + k_2) \sigma^{i3}_{(1)} + \frac{1}{3} h^3 \nabla_i \sigma^{i3}_{(2)} - \rho \left(2h + \frac{2}{3} k_1 k_2 h^3\right) \ddot{u}_3 + \\
&+ \frac{2h^3}{3} \rho (k_1 + k_2) \ddot{u}_3^{(1)} - \frac{1}{3} h^3 \rho \ddot{u}_3^{(2)} \bigg] \delta u_3 + \bigg[Q^{(1)k} - \left(2h + \frac{2}{3} k_1 k_2 h^3\right) \sigma^{k3} + \\
&+ \frac{2h^3}{3} (k_1 + k_2) \sigma^{k3}_{(1)} - \frac{2h^3}{3} \nabla_i (k_1 + k_2) \sigma^{ik} - \frac{1}{3} h^3 (\sigma^{k3}_{(2)} - 2 \nabla_i \sigma^{ik}_{(1)}) + \\
&+ \frac{2h^3}{3} \rho g^{kk} (k_1 + k_2) \ddot{u}_k - \frac{2h^3}{3} \rho g^{kk} \ddot{u}_k^{(1)} \bigg] \delta u_k^{(1)} + \\
&+ \bigg[Q^{(1)3} - \left(2h + \frac{2}{3} k_1 k_2 h^3\right) \sigma^{33} + \frac{2h^3}{3} (k_1 + k_2) \sigma^{33}_{(1)} - \\
&- \frac{2h^3}{3} \nabla_i (k_1 + k_2) \sigma^{i3} - \frac{1}{3} h^3 (\sigma^{33}_{(2)} - 2 \nabla_i \sigma^{i3}_{(1)}) + \\
&+ \frac{2h^3}{3} \rho (k_1 + k_2) \ddot{u}_3 - \frac{2h^3}{3} \rho \ddot{u}_3^{(1)} \bigg] \delta u_3^{(1)} + \bigg[Q^{(2)k} + \frac{2}{3} h^3 (k_1 + k_2) \sigma^{k3} - \\
&- \frac{1}{3} h^3 (2 \sigma^{k3}_{(1)} - \nabla_i \sigma^{ik}) - \frac{1}{3} h^3 \rho g^{kk} \ddot{u}_k \bigg] \delta u_k^{(2)} + \\
&+ \bigg[Q^{(2)3} + \frac{2}{3} h^3 (k_1 + k_2) \sigma^{33} - \frac{1}{3} h^3 (2 \sigma^{33}_{(1)} - \nabla_i \sigma^{i3}) - \frac{1}{3} h^3 \rho \ddot{u}_3 \bigg] \delta u_3^{(2)} + \\
&+ \bigg[Q^{(3)k} - \frac{1}{3} h^3 \sigma^{k3} \bigg] \delta u_k^{(3)} + \bigg[Q^{(3)3} - \frac{1}{3} h^3 \sigma^{33} \bigg] \delta u_3^{(3)} \bigg\} d\omega -
\end{aligned}
$$

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* We write out this equation, retaining terms up to the "order" h^3 inclusive, in the semi-developed form, to make the book easier to read. Of course, it is quite simple to shorten the notation here.

$$-\int\limits_{(C)}\left\{\left[\left(\left(2h+\frac{2}{3}k_1k_2h^3\right)\sigma^{ik}-\frac{2}{3}(k_1+k_2)\,h^3\sigma_{(1)}^{ik}+\frac{1}{3}\,h^3\sigma_{(2)}^{ik}\right)n_i-S^{(0)\,k}\right]\delta u_k+\right.$$

$$+\left[\left(\left(2h+\frac{2}{3}\,k_1k_2h^3\right)\sigma^{i3}-\frac{2}{3}\,(k_1+k_2)\,h^3\sigma_{(1)}^{i3}+\frac{1}{3}\,h^3\sigma_{(2)}^{i3}\right)n_i-S^{(0)\,3}\right]\delta u_3+$$

$$+\left[\left(-\frac{2}{3}\,h^3\,(k_1+k_2)\,\sigma^{ik}+\frac{2}{3}\,h^3\sigma_{(1)}^{ik}\right)n_i-S^{(1)\,k}\right]\delta u_k^{(1)}+$$

$$+\left[\left(-\frac{2}{3}\,h^3\,(k_1+k_2)\,\sigma^{i3}+\frac{2}{3}\,h^3\sigma_{(1)}^{i3}\right)n_i-S^{(1)\,3}\right]\delta u_3^{(1)}+$$

$$\left.+\left[\frac{1}{3}\,h^3\sigma^{ik}n_i-S^{(2)\,k}\right]\delta u_k^{(2)}+\left[\frac{1}{3}\,h^3\sigma^{i3}n_i-S^{(2)\,3}\right]\,\delta u_3^{(2)}\right\}ds_C=0$$

$$(i,\ k=1,\ 2). \tag{15.16}$$

where ω is the area of the basic (middle) surface, and C is the contour of the middle surface. The element of area dω and the element of arc ds_C of the contour of the middle surface, based on the relation (15.4), are expressed as follows:

$$d\omega=\sqrt{g_{11}g_{22}}\,dx^1dx^2, \tag{15.17a}$$

$$ds_C=\sqrt{g_{11}\,(\dot{x}_1)^{(2)}+g_{22}\,(\dot{x}^2)^{(2)}}\,du. \tag{15.17b}$$

The variational equation (15.16) yields a system of approximation equations for the vibrations of a shell, together with the boundary conditions.

Section 16. Differential Equations of the Oscillations of a Shell /127

Assume that the only constraints imposed on the shell are on the contour surface. Then, the variations $\delta u_i^{(n)}$ in the region ω are arbitrary independent quantities, and from their variational equation (15.16) follows the vanishing of the coefficients of these variations. Equating these coefficients to zero, we obtain the following system of differential equations:

$$\rho g^{kk}\left(2h+\frac{2}{3}\,k_1k_2h^3\right)\ddot{u}_k-\frac{2}{3}\,h^3\rho\,(k_1+k_2)\,g^{kk}\ddot{u}_k^{(1)}+\frac{1}{3}\,h^3\rho g^{kk}\ddot{u}_k^{(2)}-$$

$$-\nabla_i\left(2h+\frac{2}{3}\,k_1k_2h^3\right)\sigma^{ik}+\frac{2}{3}\,h^3\nabla_i\,(k_1+k_2)\,\sigma_{(1)}^{ik}-\frac{1}{3}\,h^3\nabla_i\sigma_{(2)}^{ik}-Q^{(0)\,k}=0; \tag{16.1a}$$

$$\rho\left(2h+\frac{2}{3}\,k_1k_2h^3\right)\ddot{u}_3-\frac{2}{3}\,h^3\rho\,(k_1+k_2)\,\ddot{u}_3^{(1)}+\frac{1}{3}\,h^3\rho\ddot{u}_3^{(2)}-$$

$$-\nabla_i\left(2h+\frac{2}{3}\,k_1k_2h^3\right)\sigma^{i3}+\frac{2h^3}{3}\,\nabla_i\,(k_1+k_2)\,\sigma_{(1)}^{i3}-\frac{1}{3}\,h^3\nabla_i\sigma_{(2)}^{i3}-Q^{(0)\,3}=0; \tag{16.1b}$$

$$\frac{2h^3}{3}\rho g^{kk}\ddot{u}_k^{(1)} - \frac{2h^3}{3}\rho g^{kk}(k_1+k_2)\ddot{u}_k + \left(2h+\frac{2}{3}k_1k_2h^3\right)\sigma^{k3} -$$
$$-\frac{2h^3}{3}(k_1+k_2)\sigma_{(1)}^{k3} + \frac{2h^3}{3}\nabla_i(k_1+k_2)\sigma^{ik} + \frac{1}{3}h^3(\sigma_{(2)}^{k3} - 2\nabla_i\sigma_{(1)}^{ik}) - Q^{(1)k} = 0; \qquad (16.2a)$$

$$\frac{2h^3}{3}\rho\ddot{u}_3^{(1)} - \frac{2h^3}{3}\rho(k_1+k_2)\ddot{u}_3 + \left(2h+\frac{2}{3}k_1k_2h^3\right)\sigma^{33} - \frac{2h^3}{3}(k_1+k_2)\sigma_{(1)}^{33} +$$
$$+\frac{2h^3}{3}\nabla_i(k_1+k_2)\sigma^{i3} + \frac{1}{3}h^3(\sigma_{(2)}^{33} - 2\nabla_i\sigma_{(1)}^{i3}) - Q^{(1)3} = 0; \qquad (16.2b)$$

$$\frac{1}{3}h^3\rho g^{kk}\ddot{u}_k - \frac{2}{3}h^3(k_1+k_2)\sigma^{k3} + \frac{1}{3}h^3(2\sigma_{(1)}^{k3} - \nabla_i\sigma^{ik}) - Q^{(2)k} = 0; \qquad (16.3a)$$
$$\frac{1}{3}h^3\rho\ddot{u}_3 - \frac{2}{3}h^3(k_1+k_2)\sigma^{33} + \frac{1}{3}h^3(2\sigma_{(1)}^{33} - \nabla_i\sigma^{i3}) - Q^{(2)3} = 0; \qquad (16.3b)$$

$$\frac{1}{3}h^3\sigma^{k3} - Q^{(3)k} = 0; \qquad (16.4a)$$

$$\frac{1}{3}h^3\sigma^{33} - Q^{(3)3} = 0 \qquad (i, k = 1, 2). \qquad (16.4b)$$

Thus, from the variational equation (15.1), with the accuracy of approximation adopted by us, we obtained a system of twelve equations which, taken together with the relations (15.12a) - (15.12c), determine twelve unknown functions $u_i^{(m)}$ (m = 0, 1, 2, 3; i = 1, 2, 3). Let us make a brief analysis of this system. /128

1. Equations (16.4a) - (16.4b) permit a direct determination of the "normal part" of the stress tensor. We know from the first version of the solution of the reduction problem that the determination of σ^{i3} (i = 1, 2, 3) is sufficient for its solution if we have recourse to the Lamé equations.

The mechanical meaning of eqs.(16.4a) - (16.4b) is that they express one of the generalizations of the Kirchhoff-Love hypothesis. In fact, if $Q^{(3)k} = Q^{(3)3} = 0$, then it follows from eqs.(16.4a) - (16.4b) that

$$\sigma^{k3} = \sigma^{33} = 0 \qquad (k = 1, 2). \qquad (16.5)$$

Of course, the relations (16.5) are less accurate than the expressions (7.5a) - (7.5b). Nevertheless, the fact of a direct connection between the Kirchhoff-Love "hypothesis" and the approximation equations of motion resulting from the variational equation (15.1) deserves attention. It may be stated that the Kirchhoff-Love "hypotheses" are a simple analytic consequence of the conditional and prescribed accuracy for the equations of the two-dimensional problem of the theory of elasticity and for the special hypotheses about the forces acting on the shell.

2. Bearing in mind eqs.(15.12a) - (15.12c), we can find the order of the system of equations set up by us, including eqs.(16.4a) - (16.4b) in this system. Equations (16.1a) - (16.3b) are equations of the second order in derivatives of the unknown functions with respect to the coordinates x^i ($i = 1, 2$) and to the time t. Equations (16.4a) - (16.4b) are equations of the first order in derivatives with respect to the coordinates. Time derivatives do not enter into these equations. Consequently, we have obtained a system of the twelfth order in derivatives with respect to the coordinates, and of the eighteenth order in derivatives with respect to time.

We recall that the system of equations (7.4a) - (7.4d) is a system of the twenty-first order and the system of equations (11.21a) - (11.21b) a system of the twelfth order. The mixed derivatives with respect to the coordinates and to time belong to the highest order with respect to the derivatives entering into eqs.(7.4a) - (7.4d). Mixed derivatives of this type do not enter into eqs.(16.1a) - (16.4b).

3. In setting up eqs.(16.1a) - (16.4b), the operation of differentiation is not performed on the components of the force vectors. Equations (7.4a) - (7.4d) are set up under the assumption that a differentiation of the components of the vectors of body forces is permissible. This gives a certain advantage to eqs.(16.1a) - (16.4b) over the equations set up according to the first version (Sect.11).

4. The difference in the composition of eqs.(16.1a) - (16.4b) and that of eqs.(7.4a) - (7.4d) can be explained by the theory of approximation functions. Equations (7.4a) - (7.4d) are set up according to one of the methods of opti- /129
mum "at-a-point" approximation functions, while eqs.(16.1a) - (16.4b) correspond to one of the methods of optimum representation of "in-the-mean" functions.

Section 17. Natural Boundary Conditions Derived from the Variational Equation (15.16)

Let us pass now to the consideration of the integral over the contour of the basic surface that enters into eq.(15.16). A study of this integral permits us to establish various versions of the boundary conditions. Here, the integral $\int_{(c)}$ vanishes if all the components under the sign of integration likewise vanish*.

For these summands to vanish, one of two conditions must be satisfied: Either the corresponding variation $\delta u_i^{(\alpha)}$ must vanish, or the coefficient of that variation must vanish. Terms in the variations δu_i^3 do not enter into the expression under the integral sign over the contour of the basic surface. Thus, the variational equation (15.16) yields nine boundary conditions, corresponding to the variations δu_i, $\delta u_i^{(1)}$, $\delta u_i^{(2)}$.

Since the order of the system of equations (16.1a) - (16.4b) relative to

* The necessity of this condition is proved in courses on the principles of analytic mechanics.

the derivatives with respect to the coordinates is twelve, it may be assumed that the system of natural boundary conditions resulting from the variational equation (15.16) is compatible with these equations. Obviously any simplification of the system of equations (16.1a) - (16.4b) must be accompanied by a change in the boundary conditions. The question of the compatibility of the boundary conditions with the system of fundamental equations must be subjected to a special analysis in specific problems.

Let us now consider, as an example, several versions of the boundary conditions.

1. With Rigidly Attached Contour Surface

In this case, we obviously have

$$u_i = u_i^{(1)} = u_i^{(2)} = 0 \qquad (i = 1, 2, 3). \tag{17.1}$$

We do not impose conditions on $u_i^{(3)}$, since the variations $\delta u_i^{(3)}$ do not enter into the integral $\int\limits_{(C)}$ and the conditions imposed on $u_i^{(3)}$ will not be natural.

2. With Free Contour Surface /130

In this case, the variations $\delta u_i^{(m)}$ (i = 1, 2, 3) are arbitrary. We obtain the natural conditions by equating the coefficients of these variations to zero. On the basis of eq.(15.16), we find

$$\left[\left(2h + \frac{2}{3} k_1 k_2 h^3\right) \sigma^{ik} - \frac{2}{3} (k_1 + k_2) h^3 \sigma^{ik}_{(1)} + \frac{1}{3} h^3 \sigma^{ik}_{(2)}\right] n_i - S^{(0)\,k} = 0; \tag{17.2a}$$

$$\left[\left(2h + \frac{2}{3} k_1 k_2 h^3\right) \sigma^{i3} - \frac{2}{3} (k_1 + k_2) h^3 \sigma^{i3}_{(1)} + \frac{1}{3} h^3 \sigma^{i3}_{(2)}\right] n_i - S^{(0)\,3} = 0; \tag{17.2b}$$

$$\left[-\frac{2}{3} h^3 (k_1 + k_2) \sigma^{ik} + \frac{2}{3} h^3 \sigma^{ik}_{(1)}\right] n_i - S^{(1)\,k} = 0; \tag{17.3a}$$

$$\left[-\frac{2}{3} h^3 (k_1 + k_2) \sigma^{i3} + \frac{2}{3} h^3 \sigma^{i3}_{(1)}\right] n_i - S^{(1)3} = 0; \tag{17.3b}$$

$$\frac{1}{3} h^3 \sigma^{ik} n_i - S^{(2)\,k} = 0; \tag{17.4a}$$

$$\frac{1}{3} h^3 \sigma^{i3} n_i - S^{(2)\,3} = 0 \qquad (i, k = 1, 2). \tag{17.4b}$$

In the other cases, the boundary conditions are mixed. Their formulation depends on the scope of the specific problems of the mechanics of shells.

Section 18. Initial Conditions

The question of the initial conditions cannot be solved by analogy to the

question of the boundary conditions from a direct study of the variational equations (15.16). If we start from considerations similar to those set forth in Sect.14, we will again obtain twenty-four initial conditions which will not be satisfied by solution of the system (16.1a) - (16.4b), since the order of this system with respect to time is eighteen. Obviously some of the generalized coordinates must obey the initial conditions, for example only the generalized coordinates u_i, $u_i^{(1)}$, $u_i^{(2)}$ ($i = 1, 2, 3$). In that case, the number of initial conditions will be equal to the order of the system.

The limitation imposed on the number of initial conditions is confirmed also by considerations based on the principles of analytical mechanics of discrete systems. A comparison of the formulation of the problems under study with the formulation of the classical problems of analytical mechanics permits us to refine the meaning of the initial condition sought.

We recall that the set of quantities determining the initial conditions in the problems of the motion of systems of material points enter into the total time derivatives which appear on the left-hand side of the general equation /131 of dynamics*. We shall perform the transformation of only one summand in the left-hand side of the general equation of dynamics (15.16). This will permit us, by analogy, to write out the required expression completely. We have

$$\begin{aligned}&\left[\rho g^{kk}\left(2h+\frac{2}{3}k_1k_2h^3\right)\ddot{u}_k-\frac{2}{3}\rho h^3(k_1+k_2)g^{kk}\ddot{u}_k^{(1)}+\right.\\&\left.+\frac{1}{3}h^3\rho g^{kk}\ddot{u}_k^{(2)}\right]\delta u_k=\frac{\partial}{\partial t}\left\{\left[\rho g^{kk}\left(2h+\frac{2}{3}k_1k_2h^3\right)\dot{u}_k-\right.\right.\\&\left.\left.-\frac{2}{3}\rho h^3(k_1+k_2)g^{kk}\dot{u}_k^{(1)}+\frac{1}{3}h^3\rho g^{kk}\dot{u}_k^{(2)}\right]\delta u_k\right\}-\\&-\left[\rho g^{kk}\left(2h+\frac{2}{3}k_1k_2h^3\right)\dot{u}_k-\frac{2}{3}\rho h^3(k_1+k_2)g^{kk}\dot{u}_k^{(1)}+\right.\\&\left.+\frac{1}{3}h\rho g^{kk}\dot{u}_k^{(2)}\right]\delta u_k.\end{aligned}\tag{a}$$

Hence we conclude that the function determining the initial conditions is of the following form:

$$\begin{aligned}\Phi=&\left[\rho g^{kk}\left(2h+\frac{2}{3}k_1k_2h^3\right)\dot{u}_k-\right.\\&\left.-\frac{2}{3}\rho h^3(k_1+k_2)g^{kk}\dot{u}_k^{(1)}+\frac{1}{3}h^3\rho g^{kk}\dot{u}_k^{(2)}\right]\delta u_k+\end{aligned}$$

* These quantities subsequently form terms outside the integral sign, which appear in the proof of the Ostrogradskiy-Hamilton principle and vanish when the paths of comparison are properly chosen.

$$
\begin{aligned}
&+\left[\rho\left(2h+\frac{2}{3}k_1k_2h^3\right)\dot{u}_3-\frac{2}{3}\rho h^3(k_1+k_2)\dot{u}_3^{(1)}+\frac{1}{3}h^3\rho\dot{u}_3^{(2)}\right]\delta u_3+\\
&\quad+\left[-\frac{2h^3}{3}\rho g^{kk}(k_1+k_2)\dot{u}_k+\frac{2h^3}{3}\rho g^{kk}\dot{u}_k^{(1)}\right]\delta u_k^{(1)}+\\
&\quad\quad+\left[-\frac{2h^3}{3}\rho(k_1+k_2)\dot{u}_3+\frac{2h^3}{3}\rho\dot{u}_3^{(1)}\right]\delta u_3^{(1)}+\\
&\quad\quad\quad+\frac{1}{3}h^3\rho g^{kk}\dot{u}_k\delta u_k^{(2)}+\frac{1}{3}h^3\rho\dot{u}_3\delta u_3^{(2)}.
\end{aligned}
\tag{18.1}
$$

The composition of the function Φ confirms our preliminary statement that the initial conditions in problems of the dynamics of shells, under the conditional accuracy of the equations here adopted, are expressed as follows:

$$
\begin{gathered}
u_{k0}^{(m)}=\varphi_k^{(m)}(x^j); \quad \dot{u}_{k0}^{(m)}=\psi_k^{(m)}(x^j) \\
(m=0,1,2; \quad j=1,2; \quad k=1,2,3).
\end{gathered}
\tag{18.2}
$$

Consequently, here too, the conditional accuracy of the equations per- /132
mits us to find the displacements only with an accuracy to terms containing the factor z^2 inclusive, although for setting up the fundamental system of equations we use terms of the form $\frac{1}{3!}z^3u_k^{(3)}$ and the equations contain terms with factors h^3.

In spite of the presence of terms with the factor h^3, it will be noted that the conditional accuracy of eqs.(16.1a) - (16.4b) in certain cases is determined by the order of the terms containing h^2. In fact, if the surface forces $X_{(\pm)i}$ vanish, then, as is obvious from the equations of generalized forces (15.15 a), the generalized forces will be of the order h. After term-by-term division of the equations by h, we obtain equations with terms containing h in a power not higher than the second.

To summarize our results, we may note that the application of the general equation of dynamics permits us to obtain better compatibility of the system of equations satisfied by the wanted functions on the basic surface with the boundary and initial conditions, than can be obtained by using the methods indicated in Sect.11.

Section 19. On Concentrated Forces

We mentioned above that the method considered in Sect.11 requires a multiple differentiability of the vector components of the active forces applied to the shell. The permissibility of this method in the case of the action of concentrated surface or body forces on the shell becomes doubtful. Of course, these doubts are connected with an object having no physical existence, namely,

the concentrated force. All the same, the concept of concentrated force, for all its abstractness, occupies a definite position among the concepts of the mathematical theory of elasticity and appears in the form of analytic singularities of the components of the displacement vector, the deformation tensor, and the stress tensor. It is therefore natural to strive for an analytically correct introduction of concentrated forces into the approximate representations of the applied theories of elasticity, and especially into the shell theory.

The theory developed in Sects.15 - 18 imposes fewer restrictions on the properties of the forces applied to the shell than the first version of the solution of the reduction problem.

The construction of eqs.(16.1a) - (16.4b) does not require that the components of body and surface forces be differentiable. These equations also apply to cases of the action of concentrated forces if the components of the concentrated forces are expressed in terms of the Dirac delta function. Hereafter, in solving eqs.(16.1a) - (16.4b) we must make use of operations that do not include the differentiation of the components of concentrated forces. It is well known that this requirement is satisfied in a number of special problems. The application of the theory of generalized functions considerably expands the class of these problems. /133

We shall also indicate a method that does not require an explicit application of the theory of generalized functions. Let us assume, for definiteness, that the concentrated force $\vec{P}$ with components P_i is applied to the boundary surface z = +h. Then, from the well-known definition of concentrated forces, we have

$$\lim_{S_\varepsilon \to 0} \iint_{(S_\varepsilon)} X_{(+)i}\, dS_\varepsilon = P_i. \tag{19.1}$$

where S_ε is the region on the boundary surface to which the point M of application of the vector $\vec{P}$ belongs.

Let us denote by $L^{(m)i}(\overset{2}{\vec{\sigma}}, \vec{u})$ the coefficients of $\delta u_i^{(m)}$ in the double integral entering into the variational equation (15.16). Obviously $L^{(m)i}$ is the differential operator defining a certain set of operations to be performed on the components of the stress tensor $\overset{2}{\vec{\sigma}}$ and the displacement vector $\vec{u}$. Let

$$\delta u_i^{(m)} = \sum_{p,q} a^{(m)}_{(pq)i} \varphi_{(pq)}(x^j), \tag{19.2}$$

where $\varphi_{(pq)}(x^j)$ is a complete system of linearly independent functions of two variables x^j on the basic surface, while the coefficients $a^{(m)}_{(pq)i}$ are arbitrary. Then, instead of the system of equations (16.1a) - (16.4b) which we obtained by equating to zero the coefficients of $\delta u_i^{(m)}$ under the sign of integration $\iint_{(\omega)}$

in eq.(15.16), we obtain an infinite system of integral relations of the following form:

$$\iint_{(\omega)} L^{(m)j}(\overset{2\to\to}{\sigma, u})\,\varphi_{(pq)}(x^j)\,d\omega + \frac{1}{m!}P_i \times$$
$$\times\left[\frac{1}{g_{ii}}\varphi_{(pq)}(x^j)(1-k_1h)(1-k_2h)h^m\right]_M = 0. \tag{19.3}$$

where the values of all quantities in brackets are taken at the point M of application of the concentrated force P.

Further determination of the required quantities from eqs.(19.3) usually leads to the solution of infinite systems of algebraic equations. The modifications of eqs.(19.3) in the case of the action of a concentrated body force are obvious.

Let us consider the case of the action of a concentrated force on the contour surface of a shell. If the concentrated force P is applied to the contour surface at the point $M(s_c^*, z^*)$, then this will introduce into the quantities $S^{(m)i}$ the following additional terms: /134

$$\Delta S^{(m)i} = \frac{P_i z^{*m}}{g_{ii} m!}\sqrt{\frac{g_{11}(1-k_1z^*)^{(2)}(\dot{x}^1)^{(2)} + g_{22}(1-k_2z^*)^{(2)}(\dot{x}^2)^{(2)}}{g_{11}(\dot{x}^1)^{(2)} + g_{22}(\dot{x}^2)^{(2)}}}\Bigg|_{s_c=s_c^*}\times$$
$$\times\,\delta(s_c - s_c^*). \tag{19.4}$$

where $\delta(s_c - s_c^*)$ is the delta function.

Let us put, on the contour surface,

$$\delta u_i^{(m)} = \sum_{(p)} b_{(p)i}\psi_{(p)}(s_c), \tag{19.5}$$

where $\psi_{(p)}(s_c)$ is a complete system of linearly independent functions of the arc s_c of the contour, while the coefficients $b_{(p)i}$ on those parts of the contour that are free from kinematic constraints are arbitrary.

Let us denote the coefficients of $\delta u_i^{(m)}$ in the integral $\int_{(C)}$ entering into the variational equation (15.16) by $M^{(m)i}(\overset{2\to\to}{\sigma, u})$. The meaning of this symbol is analogous to that of the symbol $L^{(m)i}(\overset{2\to\to}{\sigma, u})$. Then, the system of boundary conditions will lead to a system of integral relations of the form

$$\int_0^{s_c} M^{(m)\,i}(\overset{2\rightarrow}{\sigma}, \vec{u})\,\psi_{(p)}(s_c)\,ds_c = \Delta S^{(m)\,i}\,\sigma(s_c - s_c^*), \tag{19.6}$$

where $\sigma(s_c - s_c^*)$ is the Heaviside function. The integral is extended to segments of arc of the contour that are free from kinematic constraints. If the concentrated force is applied at a point where, because of kinematic considerations, we must put $\delta u_i^{(m)}$ as equal to zero, then the corresponding boundary conditions (19.6) lose their meaning.

Equations (19.6) supplement the system of conditions of the form of equation (19.3).

Section 20. Second Version of the Solution of the Problem of Reduction

In Sect.11 we pointed out that there exist two methods of solving the reduction problem if we start from the system of equations (7.4a) - (7.4d). We did not discuss the second method and confined ourselves to the statement that this method was close to the classical theory of shells. Sections 15 - 19 do not relate to the second version, since we did not refer to eqs.(7.4a) - (7.4d). Here, likewise, we shall not make use of these equations but start from the general equation of dynamics (15.1). The resultant equations will be close /135 in form to the equations of the classical theory of shells. We will therefore speak here of "the second version", although we are really going beyond the limits of the scheme given in Sect.11.

Let us return to the variational equation (15.1) and to eq.(15.10). We shall also make use of eqs.(15.5). Let us expand the quantities $\nabla_i \delta u_k^{(z)}$ in tensor series in powers of z, displacing them thus to the basic surface of the shell, and let us also displace the stress tensor components to the basic surface along the coordinate lines x^1 = const, x^2 = const, without expanding them in series, but using instead the operators of parallel displacement given previously (I, Sect.11). These components of the displaced stress tensor will be denoted by τ^{ik}. From eqs.(I, 10.1), (I, 11.12) - (I, 11.14) and (I, 11.18), we have

$$\tau^{ij} = A_p^i A_q^j \sigma^{pq} = (\delta_p^i + \Phi_p^i)(\delta_q^j + \Phi_q^j)\sigma^{pq}, \tag{a}$$

or

$$\tau^{ij} = (1 - k_i z)(1 - k_j z)\sigma^{ij}; \tag{20.1a}$$

$$\tau^{i3} = (1 - k_i z)\sigma^{i3}; \quad \tau^{33} = \sigma^{33} \tag{20.1b}$$

(i, j = 1, 2; do not sum over i and j!).

As a result we obtain from eq.(15.10), instead of eq.(15.11), the following relation:

$$\delta W^{(z)} = \tau^{ik}\nabla_i \delta u_k + \tau^{i3}(\nabla_i \delta u_3 + \delta u_i^{(1)}) + \tau^{33}\delta u_3^{(1)} +$$

$$+z[\tau^{ik}\nabla_i\delta u_k^{(1)}+\tau^{i3}(\nabla_i\delta u_3^{(1)}+\delta u_i^{(2)})+\tau^{33}\delta u_3^{(2)}]+$$
$$+\frac{1}{2}z^2[\tau^{ik}\nabla_i\delta u_k^{(2)}+\tau^{i3}(\nabla_i\delta u_3^{(2)}+\delta u_i^{(3)})+\tau^{33}\delta u_3^{(3)}]+$$
$$+\frac{1}{3!}z^3[\tau^{ik}\nabla_i\delta u_k^{(3)}+\tau^{i3}(\nabla_i\delta u_3^{(3)}+\delta u_i^{(4)})+\dot{\tau}^{33}\delta u_3^{(4)}]+\ldots \qquad (20.2)$$

Let us introduce the notation

$$T^{(m)\,ij}=\frac{1}{m!}\int_{-h}^{+h}z^m\tau^{ij}(1-k_1z)(1-k_2z)\,dz \qquad (20.3)$$

or, on the basis of eqs.(20.1a) - (20.1b),

$$T^{(m)\,ij}=\frac{1}{m!}\int_{-h}^{+h}z^m\sigma^{ij}(1-k_iz)(1-k_jz)(1-k_1z)(1-k_2z)\,dz, \qquad (20.4a)$$

$$T^{(m)\,i3}=\frac{1}{m!}\int_{-h}^{+h}z^m\sigma^{i3}(1-k_iz)(1-k_1z)(1-k_2z)\,dz, \qquad (20.4b)$$

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$$T^{(m)\,33}=\frac{1}{m!}\int_{-h}^{+h}z^m\sigma^{33}(1-k_1z)(1-k_2z)\,dz \qquad (20.4c)$$

$(m=0,1,2,3\ldots;\quad i,j=1,2;$ do not sum over i and j!).

The quantities $T^{(m)\,ij}$ are components of the second-rank tensor on the middle surface, i.e., in the set of coordinates x^j (j = 1, 2); the quantities $T^{(m)\,i3}$ are vector components in this set, and the quantity $T^{(m)\,33}$ is a scalar. From the analytic-functional viewpoint, these quantities are generalized functional moments about z, of an order different from that of the stress tensor components.

We shall now return again to the general equation of dynamics (15.1), and after transformations we reduce it to the following form:

$$\iint\limits_{(\omega)}\Big\{\Big[Q^{(0)\,k}+\nabla_iT^{(0)\,ik}-\rho g^{kk}\left(2h+\frac{2}{3}k_1k_2h^3\right)\ddot{u}_k+$$
$$+\frac{2}{3}\rho h^3(k_1+k_2)g^{kk}\ddot{u}_k^{(1)}-\frac{1}{3}h^3\rho g^{kk}\ddot{u}_k^{(2)}\Big]\delta u_k+$$

$$+\left[Q^{(0)3}+\nabla_i T^{(0)i3}-\rho\left(2h+\frac{2}{3}k_1k_2h^3\right)\ddot{u}_3+\frac{2}{3}\rho h^3(k_1+k_2)\ddot{u}_3^{(1)}-\right.$$

$$\left.-\frac{1}{3}h^3\rho\ddot{u}_3^{(2)}\right]\delta u_3+\left[Q^{(1)k}+\nabla_i T^{(1)ik}-T^{(0)k3}+\frac{2h^3}{3}\rho g^{kk}(k_1+k_2)\ddot{u}_k-\right.$$

$$\left.-\frac{2h^3}{3}\rho g^{kk}\ddot{u}_k^{(1)}\right]\delta u_k^{(1)}+\left[Q^{(1)3}+\nabla_i T^{(1)i3}-T^{(0)33}+\right.$$

$$\left.+\frac{2h^3}{3}\rho(k_1+k_2)\ddot{u}_3-\frac{2h^3}{3}\rho\ddot{u}_3^{(1)}\right]\delta u_3^{(1)}+\left[Q^{(2)k}+\nabla_i T^{(2)ik}-T^{(1)k3}-\right.$$

$$\left.-\frac{1}{3}h^3\rho g^{kk}\ddot{u}_k\right]\delta u_k^{(2)}+\left[Q^{(2)3}+\nabla_i T^{(2)i3}-T^{(1)33}-\frac{1}{3}h^3\rho\ddot{u}_3\right]\delta u_3^{(2)}+$$

$$+[Q^{(3)k}+\nabla_i T^{(3)ik}-T^{(2)k3}]\,\delta u_k^{(3)}+[Q^{(3)3}+\nabla_i T^{(3)i3}-T^{(2)33}]\,\delta u_3^{(3)}-$$

$$\left.-T^{(3)k3}\delta u_k^{(4)}-T^{(3)33}\delta u_3^{(4)}\right\}d\omega-\int_{(C)}\{[T^{(0)ik}n_i-S^{(0)k}]\,\delta u_k+$$

$$+[T^{(0)i3}n_i-S^{(0)3}]\,\delta u_3+[T^{(1)ik}n_i-S^{(1)k}]\,\delta u_k^{(1)}+$$
$$+[T^{(1)i3}n_i-S^{(1)3}]\,\delta u_3^{(1)}+[T^{(2)ik}n_i-S^{(2)k}]\,\delta u_k^{(2)}+$$
$$+[T^{(2)i3}n_i-S^{(2)3}]\,\delta u_3^{(2)}+[T^{(3)ik}n_i-S^{(3)k}]\,\delta u_k^{(3)}+$$

$$+[T^{(3)i3}n_i-S^{(3)3}]\,\delta u_3^{(3)}\}\,ds_c=0$$
$$(i,\ k=1,\ 2). \qquad (20.5)$$

The variational equation obtained here, like eq.(15.16), permits setting /137 up a two-dimensional system of differential equations of motion and to indicate the natural boundary conditions. These equations in themselves, however, do not as yet solve the reduction problem.

Section 21. First Group of Elastodynamic Equations of the Theory of Shells

Equating to zero the variational coefficients of the generalized coordinates entering into the expression under the sign of integration over the middle surface ω of the shell in eqs(20.5), we now find the following equations of motion:

$$\rho g^{kk}\left(2h+\frac{2}{3}k_1k_2h^3\right)\ddot{u}_k-\frac{2}{3}\rho h^3(k_1+k_2)g^{kk}\ddot{u}_k^{(1)}+$$
$$+\frac{1}{3}h^3\rho g^{kk}\ddot{u}_k^{(2)}-\nabla_i T^{(0)ik}-Q^{(0)k}=0; \qquad (21.1a)$$

$$\rho\left(2h+\frac{2}{3}k_1k_2h^3\right)\ddot{u}_3-\frac{2}{3}\rho h^3(k_1+k_2)\ddot{u}_3^{(1)}+\frac{1}{3}h^3\rho\ddot{u}_3^{(2)}-$$

$$- \nabla_i T^{(0)\,i3} - Q^{(0)\,3} = 0; \tag{21.1b}$$

$$\frac{2h^3}{3} \rho g^{kk} \ddot{u}_k^{(1)} - \frac{2h^3}{3} \rho g^{kk} (k_1 + k_2)\, \ddot{u}_k - \nabla_i T^{(1)\,ik} + T^{(0)\,k3} - Q^{(1)\,k} = 0; \tag{21.2a}$$

$$\frac{2h^3}{3} \rho \ddot{u}_3^{(1)} - \frac{2h^3}{3} \rho (k_1 + k_2)\, \ddot{u}_3 - \nabla_i T^{(1)\,i3} + T^{(0)\,33} - Q^{(1)\,3} = 0; \tag{21.2b}$$

$$\frac{1}{3} h^3 \rho g^{kk} \ddot{u}_k - \nabla_i T^{(2)\,ik} + T^{(1)\,k3} - Q^{(2)\,k} = 0; \tag{21.3a}$$

$$\frac{1}{3} h^3 \rho \ddot{u}_3 - \nabla_i T^{(2)\,i3} + T^{(1)\,33} - Q^{(2)\,3} = 0; \tag{21.3b}$$

$$- \nabla_i T^{(3)\,ik} + T^{(2)\,k3} - Q^{(3)\,k} = 0; \tag{21.4a}$$

$$- \nabla_i T^{(3)\,i3} + T^{(2)\,33} - Q^{(3)\,3} = 0; \tag{21.4b}$$

$$T^{(3)\,k3} = 0; \quad T^{(3)\,33} = 0 \tag{21.5}$$

$$(i,\, k = 1,\, 2).$$

The system of equations (21.1a) - (21.5) contains fifteen equations with thirty-three unknown functions. The unknowns in the equation are the twenty-four moments about z of the stress tensor components and the nine coefficients of the expansion in power series of the displacement vector components. Thus, in setting up eqs.(21.1a) - (21.5) we have been guilty of a logical incon- /138 sistency, caused by the selection of the system of generalized coordinates and leading to equations containing both moments and coefficients of an expansion in power series. This inconsistency can be eliminated by selecting the generalized coordinates in a different way. However, the system of equations (21.1a) - (21.5) obtained by the mixed method permits us to establish a direct connection with the equations of the classical theory of shells and to analyze that theory from the position of analytical mechanics.

Equations (21.4a) - (21.5) do not contain inertial terms. These equations permit finding the moments of the third and second order of the components of the normal part of the stress tensor. The meaning of equations (21.4a)-(21.5) is analogous to that of the Kirchhoff-Love hypotheses, since they make it possible to solve the reduction problem. The system of equations (21.4a) - (21.5) is indeterminate and must be supplemented by equations resulting from Hooke's law.

Section 22. Second Group of Elastodynamic Equations of the Theory of Shells

Let us express the moments of the stress tensor components in terms of the coefficients of expansions in series in powers of z of the displacement vector, making use of Hooke's law. Following the "mixed" method, we shall make use of eqs.(20.3), expanding the stress tensor components τ^{ik} into tensor power series, in ascending powers of z, thus accomplishing the parallel displacement of the stress tensor to the basic surface.

Making use of eqs.(15.12) - (15.12c), we find

$$\tau^{ik} = \sum_{n=0}^{N} \frac{z^n}{n!} [\lambda g^{ik} g^{rr} \nabla_r u_r^{(n)} + \lambda g^{ik} u_3^{(n+1)} + \mu g^{ii} g^{kk} (\nabla_i u_k^{(n)} + \nabla_k u_i^{(n)})] + \ldots, \tag{22.1a}$$

$$\tau^{i3} = \mu \sum_{n=0}^{N} \frac{z^n}{n!} g^{ii} (\nabla_i u_3^{(n)} + u_i^{(n+1)}) + \ldots, \tag{22.1b}$$

$$\tau^{33} = \sum_{n=0}^{N} \frac{z^n}{n!} [\lambda g^{rr} \nabla_r u_r^{(n)} + (\lambda + 2\mu) u_3^{(n+1)}] + \ldots \tag{22.1c}$$

(i, k = 1, 2; do not sum over i and k!).

The choice of N depends on the conditional accuracy prescribed for the equations to be set up. If the equations must not contain terms with factors h to a power higher than the third, then N ≤ 2. The number of generalized coordinates to be determined in this case is twelve. Substituting the expressions (22.1a) - (22.1c) into eqs.(20.3), we find the relations between the moments of the components of the stress tensor and the generalized coordinates. /139
We have

$$\begin{aligned} T^{(0)ik} = {} & \left(2h + \frac{2}{3} k_1 k_2 h^3\right) [\lambda g^{ik} g^{rr} \nabla_r u_r + \lambda g^{ik} u_3^{(1)} + \\ & + \mu g^{ii} g^{kk} (\nabla_i u_k + \nabla_k u_i)] - \frac{2h^3}{3} (k_1 + k_2) [\lambda g^{ik} g^{rr} \nabla_r u_r^{(1)} + \\ & + \lambda g^{ik} u_3^{(2)} + \mu g^{ii} g^{kk} (\nabla_i u_k^{(1)} + \nabla_k u_i^{(1)})] + \frac{h^3}{3} [\lambda g^{ik} g^{rr} \nabla_r u_r^{(2)} + \\ & + \lambda g^{ik} u_3^{(3)} + \mu g^{ii} g^{kk} (\nabla_i u_k^{(2)} + \nabla_k u_i^{(2)})] + \ldots, \end{aligned} \tag{22.2a}$$

$$\begin{aligned} T^{(0)i3} = \mu \Big\{ & \left(2h + \frac{2}{3} k_1 k_2 h^3\right) g^{ii} (\nabla_i u_3 + u_i^{(1)}) - \\ & - \frac{2h^3}{3} (k_1 + k_2) g^{ii} (\nabla_i u_3^{(1)} + u_i^{(2)}) + \frac{h^3}{3} g^{ii} (\nabla_i u_3^{(2)} + u_i^{(3)}) \Big\} + \ldots, \end{aligned} \tag{22.2b}$$

$$\begin{aligned} T^{(0)33} = {} & \left(2h + \frac{2}{3} k_1 k_2 h^3\right) [\lambda g^{rr} \nabla_r u_r + (\lambda + 2\mu) u_3^{(1)}] - \\ & - \frac{2h^3}{3} (k_1 + k_2) [\lambda g^{rr} \nabla_r u_r^{(1)} + (\lambda + 2\mu) u_3^{(2)}] + \\ & + \frac{h^3}{3} [\lambda g^{rr} \nabla_r u_r^{(2)} + (\lambda + 2\mu) u_3^{(3)}] + \ldots, \end{aligned} \tag{22.2c}$$

$$T^{(1)ik} = -\frac{2}{3} h^3 (k_1 + k_2) [\lambda g^{ik} g^{rr} \nabla_r u_r + \lambda g^{ik} u_3^{(1)} +$$

$$+\mu g^{ii}g^{kk}(\nabla_i u_k+\nabla_k u_i)]+\frac{2h^3}{3}[\lambda g^{ik}g^{rr}\nabla_r u_r^{(1)}+$$

$$+\lambda g^{ik}u_3^{(2)}+\mu g^{ii}g^{kk}(\nabla_i u_k^{(1)}+\nabla_k u_i^{(1)})]+\ldots, \tag{22.3a}$$

$$T^{(1)i3}=\mu\left\{-\frac{2}{3}h^3(k_1+k_2)g^{ii}(\nabla_i u_3+u_i^{(1)})+\frac{2h_3}{3}g^{ii}(\nabla^i u_3^{(1)}+u_i^{(2)})\right\}+\ldots, \tag{22.3b}$$

$$T^{(1)33}=-\frac{2}{3}h^3(k_1+k_2)[\lambda g^{rr}\nabla_r u_r+(\lambda+2\mu)u_3^{(1)}]+$$

$$+\frac{2}{3}h^3[\lambda g^{rr}\nabla_r u_r^{(1)}+(\lambda+2\mu)u_3^{(2)}]+\ldots, \tag{22.3c}$$

$$T^{(2)ik}=\frac{h^3}{3}[\lambda g^{ik}g^{rr}\nabla_r u_r+\lambda g^{ik}u_3^{(1)}+\mu g^{ii}g^{kk}(\nabla_i u_k+\nabla_k u_i)]+\ldots, \tag{22.4a}$$

$$T^{(2)i3}=\frac{\mu h^3}{3}g^{ii}(\nabla_i u_3+u_i^{(1)})+\ldots, \tag{22.4b}$$

$$T^{(2)33}=\frac{h^3}{3}[\lambda g^{rr}\nabla_r u_r+(\lambda+2\mu)u_3^{(1)}]+\ldots. \tag{22.4c}$$

At the conditional accuracy adopted here, all the moments $T^{(3)ij}$ must be /140 equated to zero:

$$T^{(3)k3}=0;\quad T^{(3)33}=0. \tag{22.5}$$

Equations (22.5) coincide with eqs.(21.5).

The systems (22.2a) - (22.5) form the second group of elastodynamic equations of the shell theory in the version given here for setting up this system. It includes eighteen equations supplementing the first group. The first and second groups together contain thirty-three equations. Obviously, under the method of calculation adopted here, the system of equations (21.1a) - (21.5) and (22.2a) - (22.5) is equivalent to the system of equations (16.1a) - (16.4b), supplemented by the relations (15.12a) - (15.12c). Therefore, we need not discuss here the general properties of the systems (21.1a) - (21.5) and (22.2a) - (22.5), since we intend to do this in our comparative analysis of the equations of the classical theory.

Section 23. Boundary and Initial Conditions

Equating to zero the components under the sign of integration over the contour of the basic surface in eq.(20.5), we obtain a system of natural boundary conditions. This system does not differ basically from the conditions considered in Sect.17. We shall, therefore, give here only the conditions on the free part of the contour surface that differ in analytic form from the conditions (17.2a) - (17.4b). The conditions on the attached part of the contour surface remain unchanged.

Thus, on the free part of the contour surface, the following conditions

are satisfied:

$$T^{(m)\,ik} n_i - S^{(m)k} = 0$$
$$(m = 0, 1, 2, 3; \quad i = 1, 2; \quad k = 1, 2, 3). \tag{23.1}$$

We shall not analyze the conditions (23.1). The main statements in Sect.17 naturally apply to this somewhat modified formulation of the boundary problem. Of course, the initial conditions considered in Sect.18 also apply to the solutions of the systems of equations (21.1a) - (21.5), (22.2a) - (22.4c).

Section 24. Generalized Conclusions and Further Development of the Analytic Mechanics of Shells

Analyzing the contents of Sect.15 - 23, we may remark that the discussed methods are based on selecting the generalized coordinates in such a manner as to restrict the number of degrees of freedom of the shell in the direction of a normal to the basic surface. The reduction of the three-dimensional problem of the theory of elasticity to the two-dimensional problem goes back to this same restriction. Similarly, the method of expansion in tensor series, a method /141 based on the use of various kinetic-geometrical hypotheses (for example, the hypotheses of straight invariant normals), in some form or other limits the number of degrees of freedom of the shell in the direction of a normal to its basic surface.

Evaluating these methods of solution of the reduction problem, we must recognize that the most logically consistent are the methods of reduction based on an application of the general equation of dynamics (15.1). These methods permit us to formulate a system of natural boundary conditions and to find initial conditions that do not explicitly contradict the properties of the solutions of the principal system of equations.

The method of expansion in series gives even less distinct grounds for establishing the system of boundary and initial conditions. There are a large number of methods of reducing the three-dimensional problems of the theory of elasticity to the two-dimensional problems of the theory of shells. All these methods are based on various selections of the system of generalized coordinates. We shall here state two choices which, in our opinion, are of fundamental interest.

1. Choice of Generalized Coordinates Corresponding to the Optimum Quadratic Approximations

We already called the reader's attention to the correlation between the reduction problem and the methods of approximate representation of functions.

The method of expansion in series is one of the methods of optimum representation of in-point functions. In the case of the presence of analytic singularities near the approximation functions, however, the process of approximation by segments of a Taylor series may prove to be divergent. This is partic-

ularly important for us, since the existence of concentrated forces acting on an elastic body cause the appearance of analytic singularities near the components of the displacement vector and the stress tensor. For this reason, we will consider a choice of generalized coordinates leading to a determination of the coefficients of the expansion of the required functions in Fourier series over the segment (-h, +h) of a normal to the basic surface of the shell, and indicate the possibilities for further development of this method. We recall that segments of a Fourier series accomplish optimum quadratic approximation to the function to be approximated*.

Let the displacement vector $\vec{u}$ undergo parallel displacement to the basic surface along a normal to that surface. Let us denote the displaced vector /142 by $\vec{v}$. From eqs.(I, 11.12) - (I, 11.14), and (I, 11.18), we have

$$v^i = u^i(1 - k_i z); \quad v^3 = u^3 \tag{24.1}$$

(i = 1, 2; do not sum over i!).

Let us expand the components v^i in Fourier series:

$$v^i = \frac{v^i_{(0)}}{2} + \sum_{m=1}^{\infty}\left(v^i_{(m)}\cos\frac{m\pi z}{h} + w^i_{(m)}\sin\frac{m\pi z}{h}\right), \tag{24.2}$$

where

$$v^i_{(m)} = \frac{1}{h}\int_{-h}^{+h} v^i \cos\frac{m\pi z}{h}\,dz; \quad w^i_{(m)} = \frac{1}{h}\int_{-h}^{+h} v^i \sin\frac{m\pi z}{h}\,dz. \tag{24.3}$$

The representation of v^i by the series (24.2) is equivalent to abandoning a determination of v^i on the boundary surface of the shell. In fact, if the series (24.2) is convergent, then, at the boundary surfaces $z = \pm h$ this series will converge to the value $\frac{1}{2}(v^i_{(+)} + v^i_{(-)})$, where $v^i_{(\pm)}$ are the values of the vector components $\vec{v}$ on the boundary surfaces of the shell.

Within the interval (-h, +h) the series will converge to the values of v^i. We therefore adopt the definition of v^i by the series (24.2) as a simplification whose meaning will be given below. The coefficients $v^i_{(m)}$ and $w^i_{(m)}$ will be considered as generalized coordinates. Further, we have

$$\delta v^i = \frac{1}{2}\delta v^i_{(0)} + \sum_{m=1}^{\infty}\left(\delta v^i_{(m)}\cos\frac{m\pi z}{h} + \delta w^i_{(m)}\sin\frac{m\pi z}{h}\right). \tag{24.4}$$

* Cf., for example, V.L.Goncharov, Theory of Interpolation and Approximation Functions, ONTI, 1934.

Let us establish the connection between the coefficient $v^i_{(m)}$, $w^i_{(m)}$ and the Fourier coefficients of the stress tensor components. Using Hooke's law, we find the following relations:

$$\tau^{ik} = \lambda g^{ik}\nabla_r v^r + \lambda g^{ik}\nabla_3 v^3 + \mu\,(g^{ii}\nabla_i v^k + g^{kk}\nabla_k v^i), \tag{24.5a}$$

$$\tau^{i3} = \mu\,(g^{ii}\nabla_i v^3 + \nabla_3 v^i), \tag{24.5b}$$

$$\tau^{33} = \lambda\nabla_r v^r + (\lambda + 2\mu)\,\nabla_3 v^3 \tag{24.5c}$$

(i, k, r = 1, 2; do not sum over i and k!).

The validity of these relations for Euclidean space within a shell is obvious*.

To find the Fourier coefficients of the stress tensor components we must /143 multiply eqs.(24.5a) - (24.5c) term by term by $\cos\frac{m\pi z}{h}$ and $\sin\frac{m\pi z}{h}$ and integrate over z from -h to +h. Let us first consider the result of this operation performed on the covariant derivative $\nabla_3 v^i$, bearing of course in mind that all the covariant derivatives in eqs.(24.5a) - (24.5c) are determined in the metric of the space adjoining the basic surface. We have

$$\nabla_3 v^i = \frac{\partial v^i}{\partial z} + \Gamma^i_{3j} v^j \overset{*}{=} \frac{\partial v^i}{\partial z} + \Gamma^i_{3i}\big|_{z=0}\, v^i, \tag{a}$$

$$\nabla_3 v^3 = \frac{\partial v^3}{\partial z} + \Gamma^3_{3j} v^j \overset{*}{=} \frac{\partial v^3}{\partial z} \tag{b}$$

(i, j = 1, 2; do not sum over i!).

Further, using (I, 11.17), we find

$$v^i_{(m)3} = \frac{1}{h}\int\limits_{-h}^{+h} \nabla_3 v^i \cos\frac{m\pi z}{h}\,dz = \frac{(v^i_{(+)} - v^i_{(-)})\cos m\pi}{h} - k_i v^i_{(m)} + \frac{m\pi}{h} w^i_{(m)} = -\,k_i v^i_{(m)} + \frac{m\pi}{h} w^i_{(m)}; \tag{24.6a}$$

$$w^i_{(m)3} = \frac{1}{h}\int\limits_{-h}^{+h} \nabla_3 v^i \sin\frac{m\pi z}{h}\,dz = -\,\frac{m\pi}{h} v^i_{(m)} - k_i w^i_{(m)} \tag{24.6b}$$

(i = 1, 2, 3; do not sum over i!).

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* In Euclidean space, an auxiliary Cartesian system can always be introduced in which the eqs.(24.5a) - (24.5c) are directly confirmed. But the tensor equations are invariant [cf. (I, Sect.6) and (Bibl.7)].

In these equations for $i = 3$, we must set $k_3 = 0$. Here we have borne in mind the above remark on the properties of the series (24.2) at $z = \pm h$. When v^i is represented by the series (24.2), the difference $v^i_+ - v^i_-$ must be taken as zero.

The covariant differentiation ∇_i for $i = 1, 2$ and integration over z are commutative, since the derivatives ∇_i are determined in the metric of a space adjoining the basic surface.

Let the expansion in a Fourier series of the stress tensor components have the following form*

$$\tau^{ik} = \frac{\tau^{ik}_{(0)}}{2} + \sum_{m=1}^{\infty}\left(\tau^{ik}_{(m)}\cos\frac{m\pi z}{h} + \theta^{ik}_{(m)}\sin\frac{m\pi z}{h}\right) \tag{24.7}$$

Then, from eqs.(24.5a) - (24.5c) and bearing eqs.(24.6a) - (24.6b) in mind, we find /144

$$\tau^{ik}_{(m)} = \lambda g^{ik}\nabla_r v^r_{(m)} + \mu\left(g^{ii}\nabla_i v^k_{(m)} + g^{kk}\nabla_k v^i_{(m)}\right) + \lambda g^{ik}\frac{m\pi}{h} w^3_{(m)}; \tag{24.8a}$$

$$\theta^{ik}_{(m)} = \lambda g^{ik}\nabla_r w^r_{(m)} + \mu\left(g^{ii}\nabla_i w^k_{(m)} + g^{kk}\nabla_k w^i_{(m)}\right) - \lambda g^{ik}\frac{m\pi}{h} v^3_{(m)}; \tag{24.8b}$$

$$\tau^{i3}_{(m)} = \mu\left[g^{ii}\nabla_i v^3_{(m)} - k_i v^i_{(m)} + \frac{m\pi}{h} w^i_{(m)}\right]; \tag{24.9a}$$

$$\theta^{i3}_{(m)} = \mu\left(g^{ii}\nabla_i w^3_{(m)} - \frac{m\pi}{h} v^i_{(m)} - k_i w^i_{(m)}\right); \tag{24.9b}$$

$$\tau^{33}_{(m)} = \lambda\nabla_r v^r_{(m)} + (\lambda + 2\mu)\frac{m\pi}{h} w^3_{(m)}; \tag{24.10a}$$

$$\theta^{33}_{(m)} = \lambda\nabla_r w^r_{(m)} - (\lambda + 2\mu)\frac{m\pi}{h} v^3_{(m)}. \tag{24.10b}$$

Equations (24.8a) - (24.10b) form one of the groups of elastodynamic equations of the shell theory. To set up the second group (equations of motion), we must again return to the general equation of dynamics. Consider first the variation $\delta W^{(z)}$ of the specific potential energy of deformation. On the basis of eq.(15.10), we find

$$\delta W^{(z)} = \tau^{ik}\nabla_i \delta v_k + \tau^{i3}\left(\nabla_i \delta v_3 + \delta\nabla_3 v_i\right) + \tau^{33}\delta\nabla_3 v_3 \quad (i, k = 1, 2). \tag{24.11}$$

* Here, too, the above statement on the series (24.2) is valid.

Since the factor $(1 - k_1 z)(1 - k_2 z)$ which, according to eq.(15.2), enters into the expression for the element of volume dV, complicates the direct transformation of the general equation of dynamics (15.1), let us put

$$V^i = (1 - k_1 z)(1 - k_2 z)\, v^i. \qquad (i = 1, 2, 3). \tag{24.12}$$

Hence, it follows that

$$\nabla_j v^i = \frac{1}{(1 - k_1 z)(1 - k_2 z)} \nabla_j V^i; \quad \nabla_3 v^i = \frac{1}{(1 - k_1 z)(1 - k_2 z)} \nabla_3 V^i + \\ + V^i \frac{\partial}{\partial z}(1 - k_1 z)^{-1}(1 - k_2 z)^{-1} \quad (i = 1, 2, 3; \quad j = 1, 2) \tag{24.13}$$

and

$$\nabla_j \delta v^i = \frac{1}{(1 - k_1 z)(1 - k_2 z)} \nabla_i \delta V^i; \quad \delta\nabla_3 v^i = \frac{1}{(1 - k_1 z)(1 - k_2 z)} \delta\nabla_3 V^i + \\ + \delta V^i \frac{\partial}{\partial z}(1 - k_1 z)^{-1}(1 - k_2 z)^{-1} \quad (i = 1, 2, 3; \quad j = 1, 2). \tag{24.14}$$

Let us represent V^i and the variations δV_i by Fourier series: /1

$$V^i = \frac{1}{2} V^i_{(0)} + \sum_{m=1}^{\infty} \left(V^i_{(m)} \cos\frac{m\pi z}{h} + W^i_{(m)} \sin\frac{m\pi z}{h} \right); \tag{24.15a}$$

$$\delta V_i \stackrel{*}{=} g_{ii} \left[\frac{1}{2} \delta V^i_{(0)} + \sum_{m=1}^{\infty} \left(\delta V^i_{(m)} \cos\frac{m\pi z}{h} + \delta W^i_{(m)} \sin\frac{m\pi z}{h} \right) \right]. \tag{24.15b}$$

Of course, the Fourier coefficients of the functions V^i and v^i as well as δV^i and δv^i are interrelated. We shall not consider these relations, but note that the arbitrariness and independence of the variations $\delta v^i_{(m)}$ and $\delta w^i_{(m)}$ results in the arbitrariness and independence of the variations $\delta V^i_{(m)}$ and $\delta W^i_{(m)}$.

Let us expand the quantities $\nabla_3 V_i$ in a Fourier series. We have

$$\nabla_3 V_i \stackrel{*}{=} g_{ii} \left[\frac{1}{2} V^i_{(0)3} + \sum_{m=1}^{\infty} \left(V^i_{(m)3} \cos\frac{m\pi z}{h} + W^i_{(m)3} \sin\frac{m\pi z}{h} \right) \right]. \tag{24.16}$$

Consequently,

$$\delta\nabla_3 V_i \stackrel{*}{=} g_{ii} \left[\frac{1}{2} \delta V^i_{(0)3} + \sum_{m=1}^{\infty} \left(\delta V^i_{(m)3} \cos\frac{m\pi z}{h} + \delta W^i_{(m)3} \sin\frac{m\pi z}{h} \right) \right]. \tag{24.17}$$

Here, from eqs.(24.6a) - (24.6b), after an obvious change of notation, we find

$$\delta V^i_{(m)3} = -k_i \delta V^i_{(m)} + \frac{m\pi}{h}\delta W^i_{(m)}; \tag{24.18a}$$

$$\delta W^i_{(m)3} = -\frac{m\pi}{h}\delta V^i_{(m)} - k_i \delta W^i_{(m)} \tag{24.18b}$$

$$(i = 1, 2, 3).$$

We recall that $k_3 = 0$.

After obvious transformations we find, from the relations (24.11) - (24.18b):

$$\begin{aligned}
&\frac{1}{h}\int_{-h}^{+h}\delta W^{(z)}(1-k_1 z)(1-k_2 z)\,dz = g_{jj}\Big\{\frac{1}{2}\nabla_i(\tau^{ij}_{(0)}\delta V^j_{(0)}) + \\
&+\sum_{m=1}^{\infty}[\nabla_i(\tau^{ij}_{(m)}\delta V^j_{(m)}) + \nabla_i(\theta^{ij}_{(m)}\delta W^j_{(m)})]\Big\} + \frac{1}{2}\nabla_i(\tau^{i3}_{(0)}\delta v^3_{(0)}) + \\
&+\sum_{m=1}^{\infty}[\nabla_i(\tau^{i3}_{(m)}\delta V^3_{(m)}) + \nabla_i(\theta^{i3}_{(m)}\delta W^3_{(m)})] + g_{jj}\Big\{\frac{1}{2}(-\nabla_i\tau^{ij}_{(0)} - k_j\tau^{j3}_{(0)} + \\
&+T^{j3}_{(0)})\,\delta V^j_{(0)} + \sum_{m=1}^{\infty}\Big[\Big(-\nabla_i\tau^{ij}_{(m)} - k_j\tau^{j3}_{(m)} - \frac{m\pi}{h}\theta^{j3}_{(m)} + T^{j3}_{(m)}\Big)\delta V^j_{(m)} + \\
&+\Big(-\nabla_i\theta^{ij}_{(m)} + \frac{m\pi}{h}\tau^{j3}_{(m)} - k_j\theta^{j3}_{(m)} + R^{j3}_{(m)}\Big)\delta W^j_{(m)}\Big]\Big\} + \frac{1}{2}(-\nabla_i\tau^{i3}_{(0)} + \\
&+T^{33}_{(0)})\,\delta V^3_{(0)} + \sum_{m=1}^{\infty}\Big[\Big(-\nabla_i\tau^{i3}_{(m)} - \frac{m\pi}{h}\theta^{33}_{(m)} + T^{33}_{(m)}\Big)\delta V^3_{(m)} + \\
&+\Big(-\nabla_i\theta^{i3}_{(m)} + \frac{m\pi}{h}\tau^{33}_{(m)} + R^{33}_{(m)}\Big)\delta W^3_{(m)}\Big] \qquad (i, j = 1, 2).
\end{aligned} \tag{24.19}$$

Here, we have introduced the notation

$$T^{i3}_{(m)} = \frac{1}{h}\int_{-h}^{+h}\tau^{i3}\,\frac{k_1 + k_2 - 2k_1k_2 z}{1-(k_1+k_2)z + k_1k_2 z^2}\cos\frac{m\pi z}{h}\,dz; \tag{24.20a}$$

$$R^{i3}_{(m)} = \frac{1}{h}\int_{-h}^{+h} \tau^{i3} \frac{k_1 + k_2 - 2k_1k_2z}{1 - (k_1 + k) z + k_1k_2z^2} \sin\frac{m\pi z}{h}\, dz \qquad (24.20b)$$

$$(i = 1, 2, 3).$$

The right-hand sides of eqs.(24.20a) - (24.20b) can be represented by series with terms expressed by $\tau^{i3}_{(n)}$ and $\theta^{i3}_{(n)}$. For sufficiently thin shells, however, there is no point in complicating the statement of the problem by considering these relations. Indeed, for thin shells the function

$$f(x^j, z) = \frac{k_1 + k_2 - 2k_1k_2z}{1 - (k_1 + k_2) z + k_1k_2z^2} \qquad (24.21)$$

is monotonous for fixed values of x^j ($j = 1, 2$) if z varies over the interval $(-h, +h)$. Then, applying the theorem on the integral mean, we find

$$T^{i3}_{(m)} = f(x^j, \zeta_1)\, \tau^{i3}_{(m)}; \qquad (24.22a)$$

$$R^{i3}_{(m)} = f(x^j, \zeta_2)\, \theta^{i3}_{(m)} \qquad (24.22b)$$

$$(i = 1, 2, 3),$$

where ζ_1 and ζ_2 are certain values of z on the interval $(-h, +h)$, which in general depend on i and m. For sufficiently thin shells, we may approximately /147 put

$$f(x^j, \zeta_i) \cong k_1 + k_2. \qquad (24.23)$$

In special cases, the function $f(x^j, z)$ is simplified. For example, for the case of plates this function vanishes.

Let us continue the transformation of the quantities entering into the variational equation (15.1). The virtual work of the forces of inertia is transformed as follows:

$$\frac{1}{h}\int_{-h}^{+h} \rho \frac{\partial^2 v_i}{\partial t^2} \delta v^i (1 - k_1 z)(1 - k_2 z)\, dz = \frac{1}{h}\int_{-h}^{+h} \rho \frac{\partial^2 v_i}{\partial t^2} \delta V^i dz =$$

$$= \rho \left[\frac{1}{2} \ddot{v}_{(0)i} \delta V^i_{(0)} + \sum_{m=1}^{\infty} (\ddot{v}_{(m)i} \delta V^i_{(m)} + \ddot{w}_{(m)i} \delta W^i_{(m)})\right]. \qquad (24.24)$$

Let us introduce the following notation for the generalized forces on the basic surface:

$$P_{(0)i} = \frac{1}{2h}\int_{-h}^{+h} \rho F_i dz + \frac{1}{2h}[X_{(+)i} + X_{(-)i}]; \tag{24.25a}$$

$$P_{(m)i} = \frac{1}{h}\int_{-h}^{+h} \rho F_i \cos\frac{m\pi z}{h}\, dz + \frac{1}{h}[X_{(+)i} + X_{(-)i}]\cos m\pi; \tag{24.25b}$$

$$Q_{(m)i} = \frac{1}{h}\int_{-h}^{+h} \rho F_i \sin\frac{m\pi z}{h}\, dz. \tag{24.25c}$$

The generalized forces on the contour surface are expressed as follows:

$$L_{(0)i} = \frac{1}{2h}\int_{-h}^{+h} X_i \varphi(z)\, dz; \quad L_{(m)i} = \frac{1}{h}\int_{-h}^{+h} X_i \varphi(z) \cos\frac{m\pi z}{h}\, dz; \tag{24.26a}$$

$$M_{(m)i} = \frac{1}{h}\int_{-h}^{+h} X_i \varphi(z) \sin\frac{m\pi z}{h}\, dz. \tag{24.26b}$$

where

$$\varphi(z) = \sqrt{\frac{g_{11}(1-k_1 z)^{(2)}(\dot{x}^1)^{(2)} + g_{22}(1-k_2 z)^{(2)}(\dot{x}^2)^{(2)}}{g_{11}(\dot{x}^1)^{(2)} + g_{22}(\dot{x}^2)^{(2)}}} \tag{24.26c}$$

Equations (24.25a) - (24.26c) are analogous in meaning to expressions (15.15a) - (15.15b).

We shall not consider the substitution of the resultant expression into /148 the general equation of dynamics (15.1) nor the simple transformations connected with such substitution, since they are analogous to those given above in Sections 15 - 23.

We shall now present the system of differential equations of motion and of the natural boundary conditions resulting from the variational equation (15.1) with our selection of generalized coordinates. The equations of motion have the following form:

$$\rho v_{(0)j} - g_{jj}(\nabla_i \tau^{ij}_{(0)} + k_j \tau^{j3}_{(0)} - T^{j3}_{(0)}) - r_{(0)j} = 0; \tag{24.27}$$

$$\rho \ddot{v}_{(m)j} - g_{jj}\left(\nabla_i \tau^{ij}_{(m)} + k_j \tau^{j3}_{(m)} + \frac{m\pi}{h}\theta^{j3}_{(m)} - T^{j3}_{(m)}\right) - P_{(m)j} = 0; \quad (24.28)$$

$$\rho \ddot{w}_{(m)j} - g_{jj}\left(\nabla_i \theta^{ij}_{(m)} - \frac{m\pi}{h}\tau^{j3}_{(m)} + k_j \theta^{j3}_{(m)} - R^{j3}_{(m)}\right) - Q_{(m)j} = 0 \quad (24.29)$$

$$(m = 1,\ 2, \ldots;\ j = 1,\ 2,\ 3;\ i = 1,\ 2; \quad \text{do not sum over } j!).$$

Here $g_{33} = 1$; $k_3 = 0$.

The natural boundary conditions on the portion of the contour surface free from kinematic constraints are:

$$\tau^{i}_{(0)\,j} n_i - L_{(0)\,j} = 0; \quad (24.30)$$

$$\tau^{i}_{(m)j} n_i - L_{(m)j} = 0; \quad (24.31)$$

$$\theta^{i}_{(m)j} n_i - M_{(m)j} = 0 \quad (24.32)$$

$$(m = 1,\ 2, \ldots;\ j = 1,\ 2,\ 3;\ i = 1,\ 2).$$

The conditions on the clamped edge will not be written out. These are obvious. Since all equations derived here are of the second order with respect to time t, the system of initial conditions does not differ from those known from Courses in the principles of mechanics.

If we make use of eqs.(24.22a) - (24.23), then the systems of equation (24.8a) - (24.10b) and (24.27) - (24.29), taken together with the boundary conditions (24.30) - (24.32), permit us to formulate autonomous boundary conditions to determine the Fourier coefficients of the wanted quantities.

These problems are all of the same type. For $m \neq 0$, each of the boundary problems leads to solution of the system of equations of the twelfth order with unknown Fourier coefficients of the displacement vector components. The solutions must satisfy six boundary conditions. For $m = 0$ the system of equations will be of the sixth order, and the number of boundary conditions will be three.

There are two additional remarks to be made on the application of Fourier series expansions to solution of the problem of reduction.

1. We have selected the segment (-h, h) as the interval of expansion. It follows from the theory of Fourier series that this interval can be extended if we indicate the analytic prolongations of the components of the displacement /149
vector beyond the segment (-h, h). Since the values of the displacement vector components v^i beyond the segment (-h, h) are arbitrary, they can be chosen such that the expansions obtained over the extended interval shall not contradict the conditions on the boundary surface of the shell.

2. The above selection of the generalized coordinates can be so modified that the operation of differentiation with respect to z of the proposed expansions of the vector components v^i shall not be explicitly nor implicitly en-

countered. For this, it is sufficient to start from the expansions of the components $\nabla_3 v^i$, and then to use the system of linear differential equations of the first order for finding the v^i. In this case, the generalized coordinates will be the quantities $v^i_{(s)\,3}$ and $w^i_{(s)\,3}$. A similar selection of generalized coordinates will be discussed below in Subsect.2.

2. One of the New Versions of the Choice of Generalized Coordinates

In most of the present Chapter we have been considering methods of reduction based on the approximate representation of the displacement vector components as functions of the coordinate z. To determine the stress tensor components as functions of z we had to differentiate the expressions of the displacements with respect to z. Such a method of reduction cannot be called optimum. This shortcoming was to some extent compensated by the use of Fourier expansions.

We shall now indicate another possible approach to elimination of this drawback. Let us approximately represent the covariant derivatives $\nabla_3 u_i$ by the equations:

$$\nabla_3 u_i \overset{*}{=} \frac{\partial u_i}{\partial z} - u_i \frac{\partial}{\partial z} \ln(1 - k_i z) = f_i(z; u_i^{(1)}, \ldots, u_i^{(n)}). \tag{24.33}$$

where f_i is a function approximately representing the derivative $\nabla_3 u_i$, while $u_i^{(1)}, \ldots, u_i^{(n)}$ are parameters considered to be generalized coordinates. They are functions of the coordinates x^j $(j = 1, 2)$ and of the time t. We recall that the quantity k_3 must be put equal to zero.

Integrating the differential equation (24.33), we find

$$u_i = u_i^{(0)}(1 - k_i z) + (1 - k_i z)\int_0^z \frac{f_i(\zeta;\, u_i^{(1)}, \ldots, u_i^{(n)})}{1 - k_i \zeta}\, d\zeta$$
$$(i = 1,\ 2,\ 3). \tag{24.34}$$

Now it is no longer necessary to differentiate with respect to z in calculating the stress tensor components.

Let us represent the function f_i by the polynomial /150

$$f_i(z,\ u_i^{(1)}, \ldots, u_i^{(n)}) = (1 - k_i z)\left(u_i^{(1)} + z u_i^{(2)} + \frac{1}{2} z^2 u_i^{(3)}\right)$$
$$(i = 1,\ 2,\ 3). \tag{24.35}$$

Then, we obtain

$$u_i \cong (1 - k_i z)\left(u_i^{(0)} + z u_i^{(1)} + \frac{1}{2} z^2 u_i^{(2)} + \frac{1}{6} z^3 u_i^{(3)}\right); \tag{24.36a}$$

$$\nabla_3 u_i \cong (1 - k_i z)\left(u_i^{(1)} + z u_i^{(2)} + \frac{1}{2} z^2 u_i^{(3)}\right)$$

$$(i = 1,\ 2,\ 3). \tag{24.36b}$$

Displacing these quantities to the basic surface, we find, from (I, 11.12) - (I, 11.13) and (I, 11.20),

$$v_i \cong u_i^{(0)} + z u_i^{(1)} + \frac{1}{2} z^2 u_i^{(2)} + \frac{1}{6} z^3 u_i^{(3)}; \tag{24.37a}$$

$$\nabla_3 v_i \cong u_i^{(1)} + z u_i^{(2)} + \frac{1}{2} z^2 u_i^{(3)}$$

$$(i = 1,\ 2,\ 3). \tag{24.37b}$$

We then set up the differential equations of motion and the boundary conditions by means of the variational equation (15.1). We leave this task to the reader as an exercise. We also call attention to the coincidence of the right-hand sides of eqs.(24.37a) - (24.37b) with the segments of the tensor series considered by us at the beginning of this Chapter. This again confirms the interrelation between the operations of expansion of tensor functions in generalized Taylor series and the parallel displacement of tensor quantities over finite distances.

Section 25. Application of Analytic Methods to the Theory of Oscillations of Layered Shells

Consider a shell consisting of parallel isotropic layers of constant nonidentical thickness. The basic surface is superposed on the boundary surface of the shell having a unit vector of the normal (I, 3.1) directed inwards in the material of the shell. This choice permits us to obtain several partial mathematical simplifications. Of course, such a selection of the basic surface is not sufficiently general, nor is it optimum. Other methods of choosing the basic surface are possible and have various advantages (Bibl.15, 21, 24). We shall not go into this question here.

In setting up the equations of motion, the conditions of connectivity /151
of the layers (II, 8.14) must be borne in mind. Of special importance here is the choice of the generalized coordinates such as to ensure maximum simplicity to the solution of the problem. We note that the conditions (II, 8.14) impose, upon the generalized coordinates, restrictions that do not depend on the law of motion, and must be considered as equations of constraint. Of the various above methods of introducing the generalized coordinates, let us discuss the method

indicated in Sect.24.2 and subject this method to a certain extension. Instead of eq.(24.33), let us set

$$\nabla_3 u_i \overset{*}{=} \frac{\partial u_i}{\partial z} - u_i \frac{\partial}{\partial z} \ln(1 - k_i z) = f_i(z;\ u_i^{(1)}, \ldots,\ u_i^{(n)}) + + (1 - k_i z) \sum_{k=1}^{m} w_i^{(k)}(x^j,\ t)\, \sigma_0 (z - z_k). \qquad (25.1)$$

where σ_0 is the Heaviside unit function, $w_i^{(k)}$ are the excess generalized coordinates introduced to satisfy the conditions of connectivity of the layers, z_k are the z-coordinates of the surfaces of contact, and m is the number of these surfaces.

Making use of eqs.(24.34) - (24.36b), we find

$$u_i \cong (1 - k_i z)\left[u_i^{(0)} + z u_i^{(1)} + \frac{1}{2} z^2 u_i^{(2)} + \frac{1}{6} z^3 u_i^{(3)}\right] + + (1 - k_i z) \int_0^z \sum_{k=1}^{m} w_i^{(k)}(x^j,\ t)\, \sigma_0 (\zeta - z_k)\, d\zeta, \qquad (25.2a)$$

$$\nabla_3 u_i \cong (1 - k_i z)\left[u_i^{(1)} + z u_i^{(2)} + \frac{1}{2} z^2 u_i^{(3)}\right] + (1 - k_j z) \times \times \sum_{k=1}^{m} w_i^{(k)}(x^j,\ t)\, \sigma_0 (z - z_k) \quad (i = 1,\ 2,\ 3). \qquad (25.2b)$$

It is clear from eq.(25.2a) that the displacement vector components and the derivatives $\nabla_i u_j$ (i = 1, 2) are continuous functions of z. The quantities $\nabla_3 u_i$ have finite discontinuities on transition across the interface of the layers. The magnitude of the discontinuity is $(1 - k_i z_k) \times w_k^{(i)}(x^j,\ t)$. We can find the quantities $w_i^{(k)}$ from the condition of continuity of the stress tensor components σ_{i3}, which we will demonstrate:

Let the z-coordinates on the boundary surfaces of the k^{th} layer be z_{k-1} and z_k; let the Lamé elastic constants of the k^{th} layer be λ_k and μ_k. Then for the $(k + 1)^{th}$ layer we have

$$\lambda_{k+1} = \lambda_k + \Delta\lambda_k; \quad \mu_{k+1} = \mu_k + \Delta\mu_k. \qquad (25.3)$$

According to Hooke's law, the stress tensor components σ_{i3} in the k^{th} and

(k + 1)th layers on their interface are expressed by the following equations: /152

$$\sigma_{i3}^{(k)}|_{z=z_k-0} = \mu_k (\nabla_i u_3 + \nabla_3 u_i)|_{z=z_k-0}; \qquad \text{(a)}$$

$$\sigma_{33}^{(k)}|_{z=z_k-0} = \lambda_k g^{ii} \nabla_i u_i|_{z=z_k-0} + (\lambda_k + 2\mu_k) \nabla_3 u_3|_{z=z_k-0}; \qquad \text{(b)}$$

$$\sigma_{i3}^{(k+1)}|_{z=z_k+0} = (\mu_k + \Gamma\mu_k) [(\nabla_i u_3 + \nabla_3 u_i)|_{z=z_k-0} + (1 - k_i z_k) w_i^{(k)}]; \qquad \text{(c)}$$

$$\sigma_{33}^{(k+1)}|_{z=z_k+0} = (\lambda_k + \Delta\lambda_k) g^{ii} \nabla_i u_i|_{z=z_k-0} + (\lambda_k + 2\mu_k + \Delta\lambda_k + + 2\Delta\mu_k) [\nabla_3 u_3|_{z=z_k-0} + w_3^{(k)}] \quad (i = 1,\ 2). \qquad \text{(d)}$$

The conditions of continuity of the stress tensor components lead to the following values for the excess coordinates $w_j^{(k)}$ ($j = 1, 2, 3$):

$$w_i^{(k)} = -\frac{\Delta\mu_k}{\mu_{k+1}} \frac{\nabla_i u_3 + \nabla_3 u_i|_{z=z_k-0}}{1 - k_i z_k}; \qquad (25.4a)$$

$$w_3^{(k)} = -\frac{1}{\lambda_{k+1} + 2\mu_{k+1}} [\Delta\lambda_k g^{ii} \nabla_i u_i + (\Delta\lambda_k + 2\Delta\mu_k) \nabla_3 u_3]|_{z=z_k-0}$$
$$(k = 1,\ 2, \ldots,\ m;\ i = 1,\ 2). \qquad (25.4b)$$

Equations (25.4a) - (25.4b) permit us to express the excess coordinates $w_j^{(k)}$ in terms of the independent coordinates $u_i^{(s)}$. As an example of the application of the analytic methods here presented, let us consider the equations of vibration of a two-layer shell.

Section 26. Equations of Oscillation of the Two-Layer Shell

Consider a shell of constant thickness 2h, consisting of two layers of respective thickness h_1 and h_2. Using eqs.(25.2a) - (25.2b) and performing a parallel displacement to the basic surface, we find, by analogy to eqs.(24.37a) - (24.37b):

$$v_i \simeq u_i^{(0)} + z u_i^{(1)} + \frac{1}{2} z^2 u_i^{(2)} + \frac{1}{6} z^3 u_i^{(3)} + \int_0^z w_i^{(1)} \sigma_0 (\zeta - h_1)\, d\zeta; \qquad (26.1a)$$

$$\nabla_3 v_i \simeq u_i^{(1)} + z u_i^{(2)} + \frac{1}{2} z^2 u_i^{(3)} + w_i^{(1)} \sigma_0 (z - h_1)$$
$$(i = 1,\ 2,\ 3). \qquad (26.1b)$$

Equations (26.1a) - (26.1b) permit us to obtain the analytic expressions for v_i and $\nabla_3 v_i$ in the first and the second layer. We have

$$v_i \simeq u_i^{(0)} + z u_i^{(1)} + \frac{1}{2} z^2 u_i^{(2)} + \frac{1}{6} z^3 u_i^{(3)} \qquad (z \leqslant h_1); \qquad (26.2a)$$

$$\nabla_3 v_i \cong u_i^{(1)} + z u_i^{(2)} + \frac{1}{2} z^2 u_i^{(3)} \qquad (z < h_1); \tag{26.2b}$$

$$v_i \cong u_i^{(0)} + z u_i^{(1)} + \frac{1}{2} z^2 u_i^{(2)} + \frac{1}{6} z^3 u_i^{(3)} + w_i^{(1)} (z - h_1) \quad (z \geqslant h_1); \tag{26.3a}$$

$$\nabla_3 v_i \cong u_i^{(1)} + z u_i^{(2)} + \frac{1}{2} z^2 u_i^{(3)} + w_i^{(1)} \qquad (z > h_1) \tag{26.3b}$$

$$(i = 1,\ 2,\ 3).$$

Let us now eliminate from these equations the excess coordinates $w_i^{(1)}$, making use of eqs.(25.4a) - (25.4b). We have

$$w_i^{(1)} \cong \mu_{12} \Big[u_i^{(1)} + \nabla_i u_3^{(0)} + h_1 (u_i^{(2)} + \nabla_i u_3^{(1)}) + \frac{1}{2} h_1^2 (u_i^{(3)} + \nabla_i u_3^{(2)}) + \\ + \frac{1}{6} h_1^3 \nabla_i u_3^{(3)} \Big]; \tag{26.4a}$$

$$w_3^{(1)} \cong \lambda_{12} g^{ii} \nabla_i u_i^{(0)} + \gamma_{12} u_3^{(1)} + h_1 [\lambda_{12} g^{ii} \nabla_i u_i^{(1)} + \gamma_{12} u_3^{(2)}] + \\ + \frac{1}{2} h_1^2 [\lambda_{12} g^{ii} \nabla_i u_i^{(2)} + \gamma_{12} u_3^{(3)}] + \frac{1}{6} h_1^3 \lambda_{12} g^{ii} \nabla_i u_i^{(3)} \tag{26.4b}$$

$$(i = 1,\ 2).$$

Here we have introduced the notation

$$\mu_{12} = -\frac{\Delta\mu_1}{\mu_2} \sigma_0 (z - h_1);\ \lambda_{12} = -\frac{\Delta\lambda_1}{\lambda_2 + 2\mu_2} \sigma_0 (z - h_1);$$

$$\gamma_{12} = -\frac{\nabla\lambda_1 + 2\Delta\mu_1}{\lambda_2 + 2\mu_2} \sigma_0 (z - h_1). \tag{26.5}$$

Further, we find

$$v_i \cong u_i^{(0)} + z u_i^{(1)} + \frac{1}{2} z^2 u_i^{(2)} + \frac{1}{6} z^3 u_i^{(3)} + \mu_{12} (z - h_1) \Big[u_i^{(1)} + \nabla_i u_3^{(0)} + \\ + h_1 (u_i^{(2)} + \nabla_i u_3^{(1)}) + \frac{1}{2} h_1^2 (u_i^{(3)} + \nabla_i u_3^{(2)}) + \frac{1}{6} h_1^3 \nabla_i u_3^{(3)} \Big]; \tag{26.6a}$$

$$v_3 \cong u_3^{(0)} + z u_3^{(1)} + \frac{1}{2} z^2 u_3^{(2)} + \frac{1}{6} z^3 u_3^{(3)} + (z - h_1) \Big[\lambda_{12} g^{ii} \nabla_i u_i^{(0)} +$$

$$+\gamma_{12}u_3^{(1)}+h_1(\lambda_{12}g^{ii}\nabla_i u_i^{(1)}+\gamma_{12}u_3^{(2)})+\frac{1}{2}h_1^2(\lambda_{12}g^{ii}\nabla_i u_i^{(2)}+$$

$$+\gamma_{12}u_3^{(3)})+\frac{1}{6}h_1^3\lambda_{12}g^{ii}\nabla_i u_i^{(3)}\Big]; \qquad (26.6b)$$

$$\nabla_3 v_i \cong u_i^{(1)}+zu_i^{(2)}+\frac{1}{2}z^2u_i^{(3)}+\mu_{12}[u_i^{(1)}+\nabla_i u_3^{(0)}+h_1(u_i^{(2)}+\nabla_i u_3^{(1)})+$$ /154

$$+\frac{1}{2}h_1^2(u_i^{(3)}+\nabla_i u_3^{(2)})+\frac{1}{6}h_1^3\nabla_i u_3^{(3)}]; \qquad (26.7a)$$

$$\nabla_3 v_3 \cong u_3^{(1)}+zu_3^{(2)}+\frac{1}{2}z^2u_3^{(3)}+\lambda_{12}g^{ii}\nabla_i u_i^{(0)}+\gamma_{12}u_3^{(1)}+h_1(\lambda_{12}g^{ii}\nabla_i u_i^{(1)}+$$

$$+\gamma_{12}u_3^{(2)})+\frac{1}{2}h_1^2(\lambda_{12}g^{ii}\nabla_i u_i^{(2)}+\gamma_{12}u_3^{(3)})+\frac{1}{6}h_1^3\lambda_{12}g^{ii}\nabla_i u_i^{(3)}$$

$$(i=1,\ 2). \qquad (26.7b)$$

In setting up these expressions we retain all terms containing the generalized coordinates $u_j^{(m)}$ ($j = 1, 2, 3$; $m = 0, 1, 2, 3$) regardless of the conventional order of smallness in these terms in connection with the presence of a factor of the form $h^i z^n$.

Let us now put eqs.(26.6a) - (26.6b) into the following form:

$$v_i \cong \sum_{m=0}^{3}\frac{1}{m!}z^m V_i^{(m)};\quad v_3 \cong \sum_{m=0}^{3}\frac{1}{m!}z^m V_3^{(m)}. \qquad (26.8)$$

where

$$V_i^{(0)} = u_i^{(0)}-\mu_{12}h_1\Big[u_i^{(1)}+\nabla_i u_3^{(0)}+h_1(u_i^{(2)}+\nabla_i u_3^{(1)})+\frac{1}{2}h_1^2(u_i^{(3)}+$$

$$+\nabla_i u_3^{(2)})+\frac{1}{3!}h_1^3\nabla_i u_3^{(3)}\Big]=u_i^{(0)}-h_1w_i^{(1)}; \qquad (26.9a)$$

$$V_3^{(0)} = u_3^{(0)}-h_1\Big[\lambda_{12}g^{ii}\nabla_i u_i^{(0)}+\gamma_{12}u_3^{(1)}+h_1(\lambda_{12}g^{ii}\nabla_i u_i^{(1)}+\gamma_{12}u_3^{(2)})+$$

$$+\frac{1}{2}h_1^2(\lambda_{12}g^{ii}\nabla_i u_i^{(2)}+\gamma_{12}u_3^{(3)})+\frac{1}{3!}h_1^3\lambda_{12}g^{ii}\nabla_i u_i^{(3)}\Big]=u_3^{(0)}-h_1w_3^{(1)}. \qquad (26.9^b)$$

(26.9b)

Further,

$$V_i^{(1)} = u_i^{(1)} + \mu_{12}\left[u_i^{(1)} + \nabla_i u_3^{(0)} + h_1(u_i^{(2)} + \nabla_i u_3^{(1)}) + \frac{1}{2}h_1^2(u_i^{(3)} + \nabla_i u_3^{(2)}) + \right.$$

$$\left. + \frac{1}{3!}h_1^3 \nabla_i u_3^{(3)}\right] = u_i^{(1)} + w_i^{(1)}; \tag{26.10a}$$

$$V_3^{(1)} = u_3^{(1)} + \lambda_{12} g^{ii} \nabla_i u_i^{(0)} + \gamma_{12} u_3^{(1)} + h_1(\lambda_{12} g^{ii} \nabla_i u_i^{(1)} + \gamma_{12} u_3^{(2)}) +$$

$$+ \frac{1}{2}h_1^2(\lambda_{12} g^{ii} \nabla_i u_i^{(2)} + \gamma_{12} u_3^{(3)}) + \frac{1}{3!}h_1^3 \gamma_{12} g^{ii} \nabla_i u_i^{(3)} = u_3^{(1)} + w_3^{(1)}; \tag{26.10b}$$

$$V_j^{(2)} = u_j^{(2)}; \quad V_j^{(3)} = u_j^{(3)} \qquad (i = 1, 2;\ j = 1, 2, 3). \tag{26.11}$$

Equations (26.7a) - (26.7b) can be represented as follows: /155

$$\nabla_3 v_i \cong \sum_{m=0}^{2} \frac{1}{m!} z^m V_i^{(m+1)}; \quad \nabla_3 v_3 \cong \sum_{m=0}^{2} \frac{1}{m!} z^m V_3^{(m+1)}. \tag{26.12}$$

Equations (26.8), together with eqs.(26.9a) - (26.10b) do not formally differ from eqs.(15.5), and the following relation obtained by expansion of the displacement vector components is a tensor series. The difference is that the quantities V_j^{0} and V_j^{1} are piecewise-continuous functions of z, which are constant on the segments of a normal to the basic surface enclosed within the layers.

Let us now find the stress tensor components, noting that in covariant differentiation with respect to x^j (j = 1, 2) the quantities z and $\sigma_0(z - h_1)$ can be considered as constants. We have

$$\tau^{ik} \cong \sum_{m=0}^{3} \frac{1}{m!} z^m \tau_{(m)}^{ik} \qquad (i,\ k = 1, 2, 3). \tag{26.13}$$

The right-hand side of eq.(26.13) does not differ in form from a segment of a Taylor tensor series, but the coefficients $\tau_{(m)}^{ik}$, except $\tau_{(m)}^{i3}$, are piecewise-continuous functions of z, constant on the segments of a normal to the basic surface enclosed between the boundary surfaces of the layer. The coefficients $\tau_{(m)}^{ik}$ are expressed by the equations

$$\tau_{(m)}^{ik} = A g^{ik} g^{ii} \nabla_i V_i^{(m)} + A g^{ik} V_3^{(m+1)} + B g^{ii} g^{kk} [\nabla_i V_k^{(m)} + \nabla_k V_i^{(m)}]; \tag{26.14a}$$

$$\tau_{(m)}^{i3} = B g^{ii} [\nabla_i V_3^{(m)} + V_i^{(m+1)}]; \tag{26.14b}$$

$$\tau^{33}_{(m)} = Ag^{rr}\nabla_r V^{(m)}_r + (A+2B)\, V^{(m+1)}_3 \tag{26.14c}$$

$(i,\ k,\ r=1,\ 2:\ m=0,\ 1,\ 2,\ 3;$ do not sum over i and k!).

Here,

$$A = \lambda_1 + \Delta\lambda_1\sigma_0 (z - h_1);\quad B = \mu_1 + \Delta\mu_1\sigma_0 (z - h_1). \tag{26.15}$$

Equations (26.14a) - (26.14c) are analogous to the relations (15.12a) - (15.12c), found for a single-layer homogeneous shell. These equations, taken together with eqs.(26.9a) - (26.10b) and (26.12), permit us to express the stress tensor components in the first and second layers in terms of the generalized coordinates $u_j^{(m)}$ and their derivatives.

Consider the variations δv_i :

$$\delta v_i = \sum_{m=0}^{3} \frac{1}{m!} z^m \delta V_i^{(m)} \tag{26.15a}$$

$$(i=1,\ 2,\ 3).$$

From eq.(26.12) there results, by analogy, /156

$$\delta \nabla_3 v_i = \sum_{m=1}^{2} \frac{1}{m!} z^m \delta V_i^{(m+1)} \tag{26.15b}$$

$$(i=1,\ 2,\ 3).$$

The variations $\delta V_i^{(0)}$ and $\delta V_i^{(1)}$ are piecewise-continuous functions of z, constant on the segments of the normal to the basic surface included within the layers. This does not permit the use of the quantities $V_i^{(m)}$ as new generalized coordinates. On the basis of eqs.(26.9a) - (26.9b) and (26.10a) - (26.10b), we have

$$\delta V_i^{(0)} = \delta u_i^{(0)} - h_1 \delta w_i^{(1)};\quad \delta V_3^{(0)} = \delta u_3^{(0)} - h_1 \delta w_3^{(1)}; \tag{26.16a}$$

$$\delta V_i^{(1)} = \delta u_i^{(1)} + \delta w_i^{(1)};\quad \delta V_3^{(1)} = \delta u_3^{(1)} + \delta w_3^{(1)} \tag{26.16b}$$

$$(i=1,\ 2).$$

where

$$\delta w_i^{(1)} = \mu_{12}\Big[\delta u_i^{(1)} + h_1 \delta u_i^{(2)} + \frac{1}{2}\, h_1^2 \delta u_i^{(3)} + \nabla_i \Big(\delta u_3^{(0)} + h_1 \delta u_3^{(1)} +$$

$$+\frac{1}{2}h_1^2\delta u_3^{(2)}+\frac{1}{3!}h_1^3\delta u_3^{(3)}\Big)\Big];\tag{26.17a}$$

$$\delta w_3^{(1)}=\gamma_{12}\left(\delta u_3^{(1)}+h_1\delta u_3^{(2)}+\frac{1}{2}h_1^2\delta u_3^{(3)}\right)+\lambda_{12}g^{rr}\nabla_r\left(\delta u_r^{(0)}+\right.$$
$$\left.+h_1\delta u_r^{(1)}+\frac{1}{2}h_1^2\delta u_r^{(2)}+\frac{1}{3!}h_1^3\delta u_r^{(3)}\right)\qquad (i,\ r=1,\ 2).\tag{26.17b}$$

The presence of covariant derivatives on the right-hand sides of equations (26.17a) - (26.17b) leads to fundamental difficulties, as will be seen from what follows.

Let us introduce notation analogous to eqs.(20.4a) - (20.4c):

$$T^{(m)ik}=\frac{1}{m!}\int_0^{2h}z_m\tau^{ik}(1-k_1z)(1-k_2z)\,dz\tag{26.18a}$$
$$(i,\ k=1,\ 2,\ 3);$$

$$H^{(m)jk}=\frac{1}{m!}\int_0^{2h}\mu_{12}z^m\tau^{jk}(1-k_1z)(1-k_2z)\,dz;\tag{26.18b}$$

$$H^{(m)i3}=\frac{1}{m!}\int_0^{2h}\gamma_{12}z^m\tau^{i3}(1-k_1z)(1-k_2z)\,dz;\tag{26.18c}$$

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$$K^{(m)I3}=\frac{1}{m!}\int_0^{2h}\lambda_{12}z^m\tau^{i3}(1-k_1z)(1-k_2z)\,dz\tag{26.18d}$$
$$(i=1,\ 2,\ 3;\ j,\ k=1,\ 2).$$

Further, let us consider the quantities connected with the inertial forces:

$$\ddot{U}^{(m)i}=\frac{1}{m!}\int_0^{2h}\rho z^m\frac{\partial^2 v^i}{\partial t^2}(1-k_1z)(1-k_2z)\,dz;\tag{26.19a}$$

$$\ddot{\Phi}^{(m)j}=\frac{1}{m!}\int_0^{2h}\rho\mu_{12}z^m\frac{\partial^2 v^j}{\partial t^2}(1-k_1z)(1-k_2z)\,dz;\tag{26.19b}$$

$$\ddot{\Phi}^{(m)3} = \frac{1}{m!}\int_0^{2h} \rho\gamma_{12} z^m \frac{\partial^2 v^3}{\partial t^2}(1-k_1 z)(1-k_2 z)\, dz; \tag{26.19c}$$

$$\ddot{\Psi}^{(m)3} = \frac{1}{m!}\int_0^{2h} \rho\lambda_{12} z^m \frac{\partial^2 v^3}{\partial t^2}(1-k_1 z)(1-k_2 z)\, dz \tag{26.19d}$$

$$(i=1,\ 2,\ 3;\ j=1,\ 2).$$

The quantities determined by eqs.(26.18a) - (26.19d) are expressed in terms of the generalized coordinates $u_j^{(m)}$, the generalized accelerations $\ddot{u}_j^{(m)}$, and their derivatives with respect to the coordinates x^j of the basic surface of the shell by means of formulas (26.8), (26.14a) - (26.15).

Let us now introduce the generalized forces. We put

$$Q^{(m)i} = \frac{1}{m!}\int_0^{2h} \rho F^i z^m (1-k_1 z)(1-k_2 z)\, dz + X^i_{(+)}(1-2k_1 h)\times$$
$$\times(1-2k_2 h)(2h)^m; \tag{26.20a}$$

$$Q^{(0)i} = \int_0^{2h} \rho F^i (1-k_1 z)(1-k_2 z)\, dz + X^i_{(+)}(1-2k_1 h)(1-2k_2 h) + X^i_{(-)}$$
$$(i=1,\ 2,\ 3;\ m=1,\ 2,\ 3); \tag{26.20b}$$

$$P^{(m)j} = \frac{1}{m!}\int_0^{2h} \mu_{12}\rho F^j z^m (1-k_1 z)(1-k_2 z)\, dz + [\mu_{12}]\, X^j_{(+)}\times$$
$$\times(1-2k_1 h)(1-2k_2 h)(2h)^m; \tag{26.21a}$$

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$$P^{(m)3} = \frac{1}{m!}\int_0^{2h} \gamma_{12}\rho F^3 z^m (1-k_1 z)(1-k_2 z)\, dz + [\gamma_{12}]\, X^3_{(+)}\times$$
$$\times(1-2k_1 h)(1-2k_2 h)(2h)^m; \tag{26.21b}$$

$$R^{(m)3} = \frac{1}{m!}\int_0^{2h} \lambda_{12}\rho F^3 z^m (1-k_1 z)(1-k_2 z)\, dz + [\lambda_{12}]\, X^3_{(+)}\times$$
$$\times(1-2k_1 h)(1-2k_2 h)(2h)^m \qquad (j=1,\ 2;\ m=0,\ 1). \tag{26.21c}$$

where $[\mu_{12}]$, $[\gamma_{12}]$ and $[\lambda_{12}]$ are the values of μ_{12}, γ_{12} and λ_{12} at $z = 2h$.

On the contour surface we determine the following quantities:

$$S^{(m)i} = \frac{1}{m!} \int_0^{2h} z^m X^i \varphi(z)\, dz, \tag{26.22a}$$

$$L^{(m)j} = \frac{1}{m!} \int_0^{2h} \mu_{12} X^j \varphi(z)\, dz, \tag{26.22b}$$

$$L^{(m)3} = \frac{1}{m!} \int_0^{2h} \gamma_{12} X^3 \varphi(z)\, dz, \tag{26.22c}$$

$$M^{(m)3} = \frac{1}{m!} \int_0^{2h} \lambda_{12} X^3 \varphi(z)\, dz, \tag{26.22d}$$

$$(i = 1,\ 2,\ 3;\ j = 1,\ 2).$$

where $\varphi(z)$ is expressed by the formula (24.26c).

Let us now consider again the general equation of dynamics (15.1), making use of eq.(24.11) in its transformation. The transformation of eq.(15.1), in this case, has two stages. The first stage leads to the following result:

$$\iint\limits_{(\omega)} \Big\{ [Q^{(0)j} + \nabla_i T^{(0)ij} - \ddot{U}^{(0)j}]\, \delta u_j^{(0)} + [Q^{(1)j} - h_1 P^{(0)j} + P^{(1)j} +$$

$$+ \nabla_i (T^{(1)ij} - h_1 H^{(0)ij} + H^{(1)ij}) - T^{(0)j3} - H^{(0)j3} - \ddot{U}^{(1)j} + h_1 \ddot{\Phi}^{(0)j} - \ddot{\Phi}^{(1)j}] \times$$

$$\times\, \delta u_j^{(1)} + [Q^{(2)j} - h_1^2 P^{(0)j} + h_1 P^{(1)j} + \nabla_i (T^{(2)ij} - h_1^2 H^{(0)ij} + h_1 H^{(1)ij}) -$$

$$- T^{(1)j3} - h_1 H^{(0)j3} - \ddot{U}^{(2)j} + h_1^2 \ddot{\Phi}^{(0)j} - h_1 \ddot{\Phi}^{(1)j}]\, \delta u_j^{(2)} + \Big[Q^{(3)j} - \frac{1}{2} h_1^3 P^{(0)j} +$$

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$$+ \frac{1}{2} h_1^2 P^{(1)j} + \nabla_i \left(T^{(3)ij} - \frac{1}{2} h_1^3 H^{(0)ij} + \frac{1}{2} h_1^2 H^{(1)ij} \right) - T^{(2)j3} -$$

$$- \frac{1}{2} h_1^2 H^{(0)j3} - \ddot{U}^{(3)j} + \frac{1}{2} h_1^3 \ddot{\Phi}^{(0)j} - \frac{1}{2} h_1^2 \ddot{\Phi}^{(1)j} \Big]\, \delta u_j^{(3)} + [- h_1 P^{(0)k} +$$

$$+ P^{(1)k} + \nabla_i (- h_1 H^{(0)ik} + H^{(1)ik}) - H^{(0)3k} + h_1 \ddot{\Phi}^{(0)k} - \ddot{\Phi}^{(1)k}] \nabla_k (\delta v_3) |_{z = h_1} +$$

$$+[-h_1R^{(0)3}+R^{(1)3}+\nabla_i(-h_1K^{(0)i3}+K^{(1)i3})-K^{(0)33}+h_1\ddot{\Psi}^{(0)3}-\ddot{\Psi}^{(1)3}]\times$$

$$\times g^{rr}\nabla_r(\delta v_r)|_{z=h_1}\Big\} d\omega+\int\limits_{(C)}\Big\{[S^{(0)j}-T^{(0)ij}n_i]\,\delta u_j^{(0)}+[S^{(1)j}-h_1L^{(0)j}+$$

$$+L^{(1)j}-(T^{(1)ij}-h_1H^{(0)ij}+H^{(1)ij})\,n_i]\,\delta u_j^{(1)}+[S^{(2)j}-h_1^2L^{(0)j}+h_1L^{(1)j}-$$

$$-(T^{(2)ij}-h_1^2H^{(0)ij}+h_1H^{(1)ij})\,n_i]\,\delta u_j^{(2)}+\Big[S^{(3)j}-\frac{1}{2}\,h_1^3L^{(0)j}+$$

$$+\frac{1}{2}\,h_1^2L^{(1)j}-\Big(T^{(3)ij}-\frac{1}{2}\,h_1^3H^{(0)ij}+\frac{1}{2}\,h_1^2H^{(1)ij}\Big)\,n_i\Big]\,\delta u_j^{(3)}+$$

$$+[-h_1L^{(0)k}+L^{(1)k}]\,\delta(\nabla_k v_3)|_{z=h_1}+[-h_1M^{(0)3}+M^{(1)3}]\times$$

$$\times g^{rr}\delta(\nabla_r v_r)|_{z=h_1}\Big\} ds_c=0$$

$$(i,\ k,\ r=1,\ 2;\ j=1,\ 2,\ 3). \tag{26.23}$$

where

$$\delta u_j|_{z=h_1}=\sum_{m=0}^{3}\frac{1}{m!}\,h_1^m\delta u_j^{(m)} \qquad (j=1,\ 2,\ 3). \tag{26.24}$$

The presence, on the left-hand side of the variational equation (26.23), of terms with the factors $g^{rr}\nabla_r(\delta v_r)|_{z=h_1}$ and $\nabla_k(\delta v_3)|_{z=h_1}$ indicates substantial differences between the cases considered earlier and this problem. These factors can be eliminated from the surface integral by means of integration by parts, i.e., by repeating the transformation of eq.(15.1), leading to equation (26.23). This is the second stage of the transformations mentioned above. The use of term-by-term integration does not require that the conditions of differentiability of the surface load components be satisfied, since this load is eliminated from the sums $-h_1P^{(0)k}+P^{(1)k}$ and $-h_1R^{(0)3}+R^{(1)3}$. The factors $g^{rr}\nabla_r(\delta v_r)|_{z=h_1}$ and $\nabla_k(\delta v_3)|_{z=h_1}$ will, however, likewise enter under the sign of integration over the contour C of the basic surface. Here, these terms cannot be excluded. Consequently, there is a substantial addition to the natural boundary conditions.

We will discuss this question later in the text, but first let us set up the system of differential equations of motion of the two-layer shell which results from eq.(26.23) after the second stage of transformations.

Section 27. Differential Equations of Motion of a Two-Layer Shell /160

We introduce the notation:

$$Y^k=-h_1P^{(0)k}+P^{(1)k}+\nabla_i(-h_1H^{(0)ik}+H^{(1)ik})-H^{(0)3k}+$$

$$+h_1\ddot{\Phi}^{(0)k}-\ddot{\Phi}^{(1)k}; \tag{27.1a}$$

$$Z = -h_1 R^{(0)3} + R^{(1)3} + \nabla_i(-h_1 K^{(0)i3} + K^{(1)i3}) - K^{(0)33} +$$
$$+ h_1 \ddot{\Psi}^{(0)3} - \ddot{\Psi}^{(1)3} \qquad (i,\ k = 1,\ 2). \tag{27.1b}$$

Performing the above transformation on the variational equation (26.23), we find by the usual method the following system of differential equations of motion:

$$\ddot{U}^{(0)k} - \nabla_i T^{(0)ik} + g^{kk}\nabla_k Z - Q^{(0)k} = 0; \tag{27.2a}$$

$$\ddot{U}^{(0)3} - \nabla_i T^{(0)i3} + \nabla_i Y^i - Q^{(0)3} = 0; \tag{27.2b}$$

$$\ddot{U}^{(1)k} - h_1\ddot{\Phi}^{(1)k} + \ddot{\Phi}^{(0)k} - \nabla_i(T^{(1)ik} - h_1 H^{(0)ik} + H^{(1)ik}) + T^{(0)k3} + H^{(0)k3} +$$
$$+ h_1 g^{kk}\nabla_k Z - Q^{(1)k} + h_1 P^{(0)k} - P^{(1)k} = 0; \tag{27.3a}$$

$$\ddot{U}^{(1)3} - h_1\ddot{\Phi}^{(0)3} + \ddot{\Phi}^{(1)3} - \nabla_i(T^{(1)i3} - h_1 H^{(0)i3} + H^{(1)i3}) + T^{(0)33} + H^{(0)33} +$$
$$+ h_1\nabla_i Y^i - Q^{(1)3} + h_1 P^{(0)3} - P^{(1)3} = 0; \tag{27.3b}$$

$$\ddot{U}^{(2)k} - h_1^2\ddot{\Phi}^{(0)k} + h_1\ddot{\Phi}^{(1)k} - \nabla_i(T^{(2)ik} - h_1^2 H^{(0)ik} + h_1 H^{(1)ik}) + T^{(1)k3} +$$
$$+ h_1 H^{(0)k3} + \frac{1}{2}h_1^2 g^{kk}\nabla_k Z - Q^{(2)k} + h_1^2 P^{(0)k} - h_1 P^{(1)k} = 0; \tag{27.4a}$$

$$\ddot{U}^{(2)3} - h_1^2\ddot{\Phi}^{(0)3} + h_1\ddot{\Phi}^{(1)3} - \nabla_i(T^{(2)i3} - h_1^2 H^{(0)i3} + h_1 H^{(1)i3}) + T^{(1)33} +$$
$$+ h_1 H^{(0)33} + \frac{1}{2}h_1^2\nabla_i Y^i - Q^{(2)3} + h_1^2 P^{(0)3} - h_1 P^{(1)3}\, 0; \quad (27.4^{b}) \tag{27.4b}$$

$$\ddot{U}^{(3)k} - \frac{1}{2}h_1^3\ddot{\Phi}^{(0)k} + \frac{1}{2}h_1^2\ddot{\Phi}^{(1)k} - \nabla_i\left(T^{(3)ik} - \frac{1}{2}h_1^3 H^{(0)ik} + \frac{1}{2}h_1^2 H^{(1)ik}\right) +$$
$$+ T^{(2)k3} + \frac{1}{2}h_1^2 H^{(0)k3} + \frac{1}{3!}h_1^3 g^{kk}\nabla_k Z - Q^{(3)k} + \frac{1}{2}h_1^3 P^{(0)k} -$$
$$-\frac{1}{2}h_1^2 P^{(1)k} = 0; \tag{27.5a}$$

$$\ddot{U}^{(3)3} - \frac{1}{2}h_1^3\ddot{\Phi}^{(0)3} + \frac{1}{2}h_1^2\ddot{\Phi}^{(1)3} - \nabla_i\left(T^{(3)i3} - \frac{1}{2}h_1^3 H^{(0)i3} + \frac{1}{2}h_1^2 H^{(1)i3}\right) +$$
$$+ T^{(2)33} + \frac{1}{2}h_1^2 H^{(0)33} + \frac{1}{3!}h_1^3\nabla_i Y^i - Q^{(3)3} + \frac{1}{2}h_1^3 P^{(0)3} -$$

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$$-\frac{1}{2}h_1^2P^{(1)3}=0$$

(i, k = 1, 2; do not sum over k!) (27.5b)

The system of equations (27.2a) - (27.5b) consists of twelve differential equations of the third order in twelve unknown functions $u_j^{(m)}$. The derivatives entering into $\nabla_k Y^k$ and $\nabla_k Z$ are of the highest order. The general order of the system is 36. Obviously, the order of the system is so high because all the terms containing the generalized coordinates introduced by us were retained in the equations, regardless of their conventional value, determined by the exponent m in the factors h^m.

Of course, such a system of equations is unsuitable for practical calculations, but it may still be used for purposes of comparison with other systems obtained by various simplifications. A qualitative analysis of eqs.(27.2a) - (27.5b) may likewise introduce new elements into the representation of the oscillatory processes in a layered shell.

Section 28. Natural Boundary Condition

Two basic forms of boundary conditions result from the variational equation (26.23).

1. With Contour Surface Kinematically not Free

On the unfree contour surface the quantities $u_j^{(m)}$ must be assigned. This assignment determines their derivatives with respect to the arc of the contour C. From the composition of the integrand expression in the integral over the contour C in eq.(26.23), it is clear that on the unfree part of a contour surface we must also prescribe the derivative, with respect to the normal to the contour, of v_3 and of the sum v_r. By this assignment, together with the assignment of the functions $u_j^{(m)}$, the covariant derivatives $\nabla_r v_r |_{z=h_1}$, and $\nabla_k v_3 |_{z=h_1}$, are determined on the contour C. The total number of boundary conditions here is not single-valued.

In fact, we may prescribe all the derivatives with respect to the normal to the contour C of the quantities $u_j^{(m)}$. In that case, we will obtain twenty-four boundary conditions. This number of boundary conditions does not corre- /162 spond to the order of the system (27.2a) - (27.5b).

If we directly prescribe the normal derivatives of $g^{rr} v_r^{\cdot} |_{z=h_1}$ and $v_3 |_{z=h_1}$, then we will have fourteen boundary conditions. The uniqueness of the solution of the boundary problem in this case requires further investigation.

2. With the Contour Surface Free

If the boundary surfaces of the shell are free from kinematic constraint, then the variational field of $\delta u_j^{(m)}$ is entirely arbitrary within the region ω and along its boundary. An arbitrary variational field of $\delta u_j^{(m)}$ may also be repre-

sented beyond the boundary of the region ω. This shows that on the boundary the variation $g^{rr}\delta\nabla_r v_r|_{z=h_1}$ must be regarded as an arbitrary, independent quantity.

In assigning the variations $\delta u_j^{(m)}$ on the contour surface, the derivatives $\delta\nabla_k v_3$ will be bound by a linear relation resulting from the expression of the derivatives of δv_3 over the arc of the contour C. For this reason, fourteen boundary conditions can be obtained from eq.(26.23):

$$S^{(0)k} - T^{(0)ik}n_i + g^{kk}Zn_k = 0; \tag{28.1a}$$

$$S^{(0)3} - (T^{(0)i3} - Y^i)\,n_i + N^{(0)3} = 0; \tag{28.1b}$$

$$S^{(1)k} - h_1L^{(0)k} + L^{(1)k} - (T^{(1)ik} - h_1H^{(0)ik} + H^{(1)ik})\,n_i + h_1g^{kk}Zn_k = 0; \tag{28.2a}$$

$$S^{(1)3} - h_1L^{(0)3} + L^{(1)3} - (T^{(1)i3} - h_1H^{(0)i3} + H^{(1)i3} - h_1Y^i)\,n_i + N^{(1)3} = 0; \tag{28.2b}$$

$$\begin{aligned} S^{(2)k} - h_1^2L^{(0)k} + h_1L^{(1)k} - (T^{(2)ik} - h_1^2H^{(0)ik} + h_1H^{(1)ik})\,n_i + \\ + \frac{1}{2}\,h_1^2g^{kk}Zn_k = 0; \end{aligned} \tag{28.3a}$$

$$\begin{aligned} S^{(2)3} - h_1^2L^{(0)3} + h_1L^{(1)3} - \left(T^{(2)i3} - h_1^2H^{(0)i3} + h_1H^{(1)i3} - \frac{1}{2}\,h_1^2Y^i\right)n_i + \\ + N^{(2)3} = 0; \end{aligned} \tag{28.3b}$$

$$\begin{aligned} S^{(3)k} - \frac{1}{2}\,h_1^3L^{(0)k} + \frac{1}{2}\,h_1^2L^{(1)k} - \left(T^{(3)ik} - \frac{1}{2}\,h_1^3H^{(0)ik} + \frac{1}{2}\,h_1^2H^{(1)ik}\right)n_i + \\ + \frac{1}{3!}\,h_1^3g^{kk}Zn_k + 0; \end{aligned} \tag{28.4a}$$

$$\begin{aligned} S^{(3)3} - \frac{1}{2}\,h_1^3L^{(0)3} + \frac{1}{2}\,h_1^2L^{(1)3} - \left(T^{(3)i3} - \frac{1}{2}\,h_1^3H^{(0)i3} + \frac{1}{2}\,h_1^2H^{(1)i3} - \right. \\ \left. - \frac{1}{3!}\,h_1^3Y^i\right)n_i + N^{(3)3} = 0; \end{aligned} \tag{28.4b}$$

$$A_k\,[-h_1L^{(0)k} + L^{(1)k}] = 0; \tag{28.5}$$

$$-h_1M^{(0)3} + M^{(1)3} = 0 \tag{28.6}$$

(i, k = 1, 2; do not sum over k!).

The coefficients A_k are functions of the arc of the contour C. They are /163 determined after elimination of one of the derivatives $\nabla_k\delta v_3|_{z=h}$ by means of the expression of the absolute derivative of $\delta v_3|_{z=h}$ over the arc of the contour s. The quantities $N^{(m)}{}_3$ are derived from the term-by-term integration of the summand under the integration sign $\int_{(C)}$ in eq.(26.23), containing the absolute derivative of $\delta v_3|_{z=h_1}$ over the arc s. This summand also results from elimination of one of the derivatives $\nabla_k\delta v_3|_{z=h_1}$, as already mentioned. We will not give the details on these calculations, but rather pass to brief conclusions generalizing the various methods of reduction of the three-dimensional problems

of the elasticity theory to two-dimensional problems.

Section 29. Classical Theory of Shells

The investigation made by us naturally includes a brief analysis of the classical theory of shells and of certain works that have expanded the field of this theory. We will discuss the characterization of the analytic properties of the fundamental quantities which are the object of investigation in the classical theory, and on the methods of investigation.

1. Forces and Moments

In the Kirchhoff-Love theory of shells, the stress tensor is replaced by a system of forces and moments determining the principal vector and the principal moment of the internal forces in the shell, reduced to a point lying on the contour of an element of the middle surface and referred to unit length of the corresponding coordinate line.

It is easy to convince ourselves that the components of the forces and moments so determined are not components of vectors obeying the rules of transformation of tensor quantities. The forces and moments can evidently be connected with the vector components of the stress tensor (Bibl.8). We shall not go into details on this approach.

After constructing the vectors of the stresses acting on the contour surfaces of an element of the shell, we will subject them to parallel displacement to the basic surface, on the basis of (I, 11.18), and will then relate them to unit length of the corresponding coordinate line. We obtain

$$T^i_{(k)} = \sqrt{g_{kk}} \int_{-h}^{+h} \sigma^{ik} (1 - zk_i)(1 - zk_1)(1 - zk_2)\, dz; \qquad (29.1a)$$

$$T^3_{(k)} = \sqrt{g_{kk}} \int_{-h}^{+h} \sigma^{3k} (1 - zk_1)(1 - zk_2)\, dz \qquad (i, k = 1, 2). \qquad (29.1b)$$

Here the index (k) indicates the number of the coordinate line normal to that /164
part of the contour surface of the shell element on which the stress vector acts, yielding the quantities $T^1_{(k)}$ and $T^3_{(k)}$. The other notation is conventional.

Further, making use of (I, 8.6), which defines the covariant components of a vector product, we find the components of the moments of the annexed couples, produced when the stress vectors are reduced to the basic surface, referring them to unit length of the corresponding coordinate line. Neglecting all terms that are nonlinear with respect to the strain tensor components, we find

$$M_{(1)1} = -g_{11} \sqrt{g_{22}} \int_{-h}^{+h} \int_{0}^{z} \sigma^{21} (1 - zk_1)(1 - zk_2)^2\, dx^3\, dz =$$

$$= - g_{11}\sqrt{g_{22}}\int\limits_{-h}^{+h} z\sigma^{21}(1 - zk_1)(1 - zk_2)^2\,dz; \tag{29.2a}$$

$$M_{(1)2} = g_{11}\sqrt{g_{22}}\int\limits_{-h}^{+h} z\sigma^{11}(1 - zk_1)^2(1 - zk_2)\,dz; \tag{29.2b}$$

$$M_{(1)3} = 0; \tag{29.2c}$$

$$M_{(2)1} = - g_{22}\sqrt{g_{11}}\int\limits_{-h}^{+h} z\sigma^{22}(1 - zk_1)(1 - zk_2)^2\,dz; \tag{29.2d}$$

$$M_{(2)2} = g_{22}\sqrt{g_{11}}\int\limits_{-h}^{+h} z\sigma^{12}(1 - zk_1)^2(1 - zk_2)\,dz; \tag{29.2e}$$

$$M_{(2)3} = 0. \tag{29.2f}$$

Equations (29.1a) - (29.2f) differ in form from those familiar from the classical theory of shells*. But this difference is not substantial, since most works on shell theory replace the components of the tensor quantities by their "physical components", which may be obtained by using eqs.(I, 5.20) - (I, 5.2) and noting the remark in (I, Sect.7). We have, for example,

$$\sigma^{11} = \frac{\sigma_{x^1x^1}}{\overset{z}{g}_{11}} = \frac{\sigma_{x^1x^1}}{g_{11}}(1 - zk_1)^{-2}; \tag{a}$$

and also

$$T^1_{(1)} = \frac{T_{(1)x^1}}{\sqrt{g_{11}}}. \tag{b}$$

Substituting the expressions (a) and (b) into one of eqs.(29.1a), we find /165

$$T_{(1)x^1} = \int\limits_{-h}^{+h} \sigma_{x^1x^1}(1 - zk_2)\,dz. \tag{c}$$

The relation (c) is known from the classical theory. By analogy, putting

$$\sigma^{21} = \frac{\sigma_{x^1x^2}}{\sqrt{\overset{z}{g}_{11}\overset{z}{g}_{22}}} = \sigma_{x^1x^2}(g_{11}g_{22})^{-\frac{1}{2}}(1 - zk_1)^{-1}(1 - zk_2)^{-1}, \tag{d}$$

* Cf. Arthur Love, The Mathematical Theory of Elasticity, ONTI, 1935, or (Bibl.5,11).

we find from eq.(29.2a):

$$M_{(1)x^1} = \frac{M_{(1)1}}{\sqrt{g_{11}}} = -\int_{-h}^{+h} z\sigma_{x^1x^2}(1 - zk_2)\,dz. \tag{e}$$

This is the well-known expression for the torsional moment. Thus, the method employed here, based on the theory of parallel displacement of tensor quantities, leads to the results of the classical theory if we remain within the limits of the linear theory of elasticity. If we retain the nonlinear terms, however, this method leads instead to results that refine the classical theory. as shown by us elsewhere (Bibl.23a, b). We shall not consider the nonlinear theory here.

We will now discuss the connection between the forces, moments, and the quantities $T^{(m)ik}$ as determined by eqs.(20.4a) - (20.4c). The quantities $T^{(0)ij}$ ($i, j = 1, 2$) are connected by linear relations with the forces and moments. In fact, eq.(20.4a) yields

$$T^{(0)ij} = \int_{-h}^{+h} \sigma^{ij}(1 - k_i z)(1 - k_1 z)(1 - k_2 z)\,dz -$$

$$- k_j \int_{-h}^{+h} z\sigma^{ij}(1 - k_i z)(1 - k_1 z)(1 - k_2 z)\,dz. \tag{f}$$

From this the above-mentioned relation is obtained.

The quantities $T^{(1)ij}$ cannot be expressed in terms of the forces and moments. This also applied to the quantities $T^{(m)i3}$ and $T^{(m)33}$. We note that only in the approximate theory which contains errors that permit neglecting all terms of the order of hk_i, are the quantities $T^{(0)ij}$ and $T^{(0)i3}$ proportional to the forces, and the quantities $T^{(1)ij}$ proportional to the moments.

All the above again leads to the conclusion that the approximate replacement of the stress tensor by a system of forces and moments is justified only in the case of a preliminary introduction of simplifying hypotheses analogous to those of Kirchhoff-Love. For this reason, various generalizations of the classical theory involving the use of reduced forces and moments are of lim- /166
ited value. In particular, the nonlinear theory of plates and shells, constructed with the use of reduced forces and moments, contains errors that decrease the significance of the introduction of the nonlinear terms.

2. Equations of Equilibrium and Motion

The shortcomings connected with the description of the stressed state of a shell element by a system of forces and moments are especially pronounced when we consider the equations of motion or, in particular, the equations of equilibrium of this element. We recall that an element of a shell has a finite dimension in the direction of the line x^3. The equations of motion of an element of the shell in the classical theory are set up as the equations of a rigid body. They result from the theorem on the motion of the center of inertia of a shell element and the theorem on the variation of its kinetic moment. Clearly such an approach to setting up the equations of motion is based on the preliminary application of one of two methods of reduction of the three-dimensional problems of the theory of elasticity to two-dimensional problems; this is the method based on the application of the Kirchhoff-Love hypotheses,or expansions in tensor series followed by elimination of the derivatives $\nabla_3 \ldots \nabla_3 u_i$. Essentially, the use of the equations of motion of an element "as a whole", by defining the general statement of the reduction problem, permits exclusion of eqs.(7.4b) and (7.4d) from consideration. We have mentioned this fact in Sect.7 and in the subsequent discussion.

We shall not here consider all the classical equations of motion of a shell element, but rather focus our attention on the sixth equation, containing the component of the moment of external forces M^3, referred to unit area of the basic surface of the shell.

Assume that the deformed shell is referred to the system of coordinates x^i. The coordinate lines x^1 and x^2 on the deformed basic surface coincide with its lines of curvature, while the vector e_3 of the coordinate basis is directed along the normal to it, and is equal, modulo, to unity. Then, we may make use of eqs.(29.1a) - (29.2f), but we must remember that all the quantities entering into them relate to the deformed shell.

Following our other work (Bibl.23a, b), we shall show that the sixth equation of equilibrium is not satisfied if the component M^3 of the principal moment of external forces does not vanish. In connection with the vanishing of $M_{(1)3}$ and $M_{(2)3}$ in these formulas (29.2c) and (29.2f), the sixth equation of equilibrium in the system of coordinate selected by us has the following form (see Bibl.23a, b):

$$\Gamma^3_{11} M^1_{(1)} \sqrt{g_{22}} + \Gamma^3_{22} M^2_2 \sqrt{g_{11}} + \frac{1}{\sqrt{g}} [g_{11} T_{(1)2} \sqrt{g_{22}} - - g_{22} T_{(2)1} \sqrt{g_{11}}] + M^3 = 0. \tag{29.3}$$

Making use of eqs.(29.1a) - (29.2f), we find /167

$$M^1_{(1)} = \frac{1}{g_{11}} M_{(1)1} = - \sqrt{g_{22}} \int_{-h_1}^{h_2} z\sigma^{12} (1 - zk_1)(1 - zk_2)^2 \, dz; \tag{g}$$

$$M^2_{(2)} = \sqrt{g_{11}} \int_{-h_1}^{h_2} z\sigma^{12} (1 - zk_1)^2 (1 - zk_2)\, dz; \tag{h}$$

$$T_{(1)2} = g_{22} T^2_{(1)} = g_{22} \sqrt{g_{11}} \int_{-h_1}^{h_2} \sigma^{12} (1 - zk_1)(1 - zk_2)^2\, dz; \tag{i}$$

$$T_{(2)1} = g_{11} T^1_{(2)} = g_{11} \sqrt{g_{22}} \int_{-h_1}^{h_2} \sigma^{12} (1 - zk_1)^2 (1 - zk_2)\, dz. \tag{k}$$

In these equations, h_1 and h_2 are the distances along the normal from the deformed basic surface to the boundary surfaces.

Further, let us use eqs.(4.4):

$$\Gamma^3_{11} = g_{11} k_1; \quad \Gamma^3_{22} = g_{22} k_2. \tag{ℓ}$$

Substituting into eq.(29.3) the relation (g) - (ℓ), we find

$$\begin{aligned}
&\Gamma^3_{11} M^1_{(1)} \sqrt{g_{22}} + \Gamma^3_{22} M^2_{(2)} \sqrt{g_{11}} + \frac{1}{\sqrt{g}} [g_{11} T_{(1)2} \sqrt{g_{22}} - g_{22} T_{(2)1} \sqrt{g_{11}}] = \\
&= g_{11} g_{22} \Bigg[- \int_{-h_1}^{h_2} zk_1 \sigma^{12} (1 - zk_1)(1 - zk_2)^2\, dz + \int_{-h_1}^{h_2} zk_2 \sigma^{12} (1 - zk_1)^2 \times \\
&\times (1 - zk_2)\, dz + \int_{-h_1}^{h_2} \sigma^{12} (1 - zk_1)(1 - zk_2)^2\, dz - \int_{-h_1}^{h_2} \sigma^{12} (1 - zk_1)^2 \times \\
&\times (1 - zk_2)\, dz \Bigg] = g_{11} g_{22} \Bigg[\int_{-h_1}^{h_2} \sigma^{12} (1 - zk_1)^2 (1 - zk_2)^2\, dz - \int_{-h_1}^{h_2} \sigma^{12} \times \\
&\times (1 - zk_1)^2 (1 - zk_2)^2\, dz \Bigg] \equiv 0.
\end{aligned} \tag{m}$$

Consequently, eq.(29.3) reduces to the condition

$$M^3 = 0. \tag{29.4}$$

Thus, the system of forces and moments (29.1a) - (29.2f) reduced to the basic surface, cannot balance the external forces, if they are reduced to a /168

couple lying in a plane parallel to the plane of a tangent to the basic surface*.

We shall now make two remarks on the result.

1. We have used a special selection of the coordinate system connected with the deformed shell. However, the invariant properties of tensor equations of equilibrium permit us to assert that the result is valid in any system (I, Sect.6).

Of course, if the choice of the coordinate system is arbitrary, we will not obtain eqs.(29.4). But in this case, only two of the three equations of equilibrium containing moments of internal forces will be independent.

2. If the component M^3 vanishes, then the sixth equation of equilibrium is satisfied identically only on the basis of the expressions of forces and moments (29.1a) - (29.2f). The identical satisfaction of this equation is entirely unconnected with the relations between forces, moments, components of the strain tensor of the basic surface, and the tensor of variation of its curvature (Section 10) resulting from Hooke's law. For this reason, the identical satisfaction of the sixth equation of equilibrium by relations resulting from equations approximately expressing Hooke's law must be considered only as an indication that the approximation adopted can in fact be satisfied.

Returning to the introductory remarks on the equations of equilibrium and the motion of the shell, we note that the condition (29.4) imposed on the external forces reveals the insufficiency of the description of the stressed state of the shell by a system of forces and moments reduced to the basic surface. This insufficiency has no effect on the solutions of most technical problems of the shell theory, since in these problems the condition (29.4) is usually satisfied.

Section 30. Brief Survey of Recent Results of Reducing the Three-Dimensional Problem of the Theory of Elasticity to the Two-Dimensional Problem of the Theory of Shells

In conclusion, let us give a characterization of the results obtained in solving the reduction problem during the last quarter century. Here, we will not analyze the outstanding work by F.Krauss written in 1929** but merely remark that he posed the problem of constructing a statics for shells that did not rely on the Kirchhoff-Love hypotheses.

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* This was first established by a different method by F.Krauss in his paper Fundamental Equations of Shell Theory, Math. Ann., Vol.101, 1929. This proof was mentioned by us elsewhere (Bibl.23a) in 1938. See also the monograph by V.Z.Vlasov (Bibl.3a).

** See preceding footnote. A brief analysis of Krauss' investigations will be found elsewhere (Bibl.23b).

1. Reduction by the Use of Series. Application of the D'Alembert-Lagrange Principle /169

In 1938 - 1940 (Bibl.23a, b) one version of the analytic statics of shells was studied, based on the use of expansions of stress and strain tensor components in MacLaurin tensor series in powers of the coordinate $x^3 = z$. This method, with the necessary general equations, is given at the beginning of this Chapter for dynamic problems.

The reduction method based on the expansion of the wanted quantity in power series of z was applied to problems of the statics of shells and plates by A.I.Lur'ye in 1940 - 1942 (Bibl.25a, b). In one paper (Bibl.25b), he studied the equilibrium of a plane plate and showed that the displacement of any point of the plate could be expressed in terms of certain functions determined by the loads on the faces of the plate (for $z = \pm h$) in the form of series for which the form of the n^{th} term was established. This same method was employed in a monograph (Bibl.9b) in studying the equilibrium of a plane layer. The results were obtained by the symbolic method. The work of A.I.Lur'ye confirms the significance of the reduction method based on an expansion in power series of z.

We find the idea of the combined use of the general equation of statics and expansion in power series of z of the stress and strain tensor components, with the object of solving the reduction problem (Bibl.3a)*. Here the hypotheses of Kirchhoff-Love are used, and the shell element is regarded as an absolutely rigid body with six degrees of freedom. Clearly, under these assumptions, the application of the general equation of statics introduces no substantially new elements into the solution of the problem of reducing the three-dimensional problem of elasticity theory to a two-dimensional problem, and as a result we obtain the equations of the classical statics of shells.

This method was further developed by Kh.M.Mushtary and I.G.Teregulov (Bibl.27), who studied the reduction problem for the static problem in nonlinear formulation, using expansions of the displacement vector components in series in powers of the variable $x^3 = z$. We used a similar device in the linear formulation in Sect.15 - 23, when we investigated the problems of elastodynamics.

The general equation of dynamics (15.1) holds latent possibilities for the development of a reduction theory. Certain applications of this equation to the new formulations of the dynamic boundary problems of shell theory have already been indicated by us in Sect.28**. Of course, even these results do not exhaust all the facts obtainable from eq.(15.1). /170

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* The general equation of dynamics was evidently applied by A.Basset to construction of the equations of the classical theory of shells, for cylindrical and spherical shells, as far back as 1890. See A.Love, Mathematical Theory of Elasticity, ONTI, 1935, pp.559-561.

** The reader will find several data on the development of investigations on the reduction problem in another paper (Bibl.26).

Methods of reduction based on the use of series expansions have been developed as early as 1942 by Epstein, Kennard, and others who studied the dynamics of cylindrical shells and were apparently unacquainted with the work of Soviet scientists*. Beginning from about 1948, these generalizations, mainly of the dynamics of plates and shells, became widespread everywhere. The cause of this new interest in studies which certain scientists formerly classified as theoretical investigations without practical value, was the need to establish a dynamics of plates and shells suitable for the study of various high-frequency vibrations and transients of dynamic loading.

It must be mentioned again that priority in the development of the generalizations of the theory of plates and shells belongs to Ukrainian and Soviet scientists**.

2. The "Semi-Inverse" Method of Reduction

During the last decade, a new trend has developed in the methods of reduction of the three-dimensional problems of the theory of elasticity to the two-dimensional problems of the mechanics of plates and shells. These methods may be called "semi-inverse", since their distinctive feature is the preliminary determination of certain components of the stress or strain tensor by certain functions of the coordinate $x^3 = z$.

In chronological order, in this respect, we must list the work of E.Reissner on the theory of equilibrium of thin plates***. An analysis of Reissner's work and its possible generalization is given in another paper (Bibl.20a). /171 Reissner expressed components of that part of the stress tensor tangential to the middle plane by linear functions of the coordinate z, and determined the components of the normal part from the equations of equilibrium, finding the indeterminate elements of the solution from the conditions on the boundary surfaces****. He thus obtained a solution satisfying the conditions of equili-

* a) P.S.Epstein, On the Theory of Elastic Vibrations in Plates and Shells. J. Math. and Phys., Vol. 21, 1942
b) E.H.Kennard, The New Approach to Shell Theory: Circular Cylinders. IAM, Vol.20, No.1, 1953; Cylindrical Shells: Energy, Equilibrium, Addenda and Erratum. IAM, Vol.22, No.1, 1955; Approximate Energy and Equilibrium Equations for Cylindrical Shells. IAM, Vol.23, No.4, 1956; A Fresh Test of the Epstein Equations for Cylinders. IAM, Vol.25, No.4, 1958. See also USSR Abstract Journal of Mechanics, No.2, Abstract No.802, 1953

** The generalized equation for transverse vibrations of rods with allowance for the effect of shear and inertia of rotation was found by S.P.Timoshenko in 1921-1922. These results were extended to the theory of vibrations of plates by Ya.S.Uflyand in his paper "Propagation of Waves in Transverse Vibrations of Rods and Plates". PMM, Vol.XII, No.3, 1948

*** E.Reissner, a) On the Theory of Bending of Elastic Plates. J. Math. And Phys., Vol.XXIII, 1944. b) On Bending of Elastic Plates. Quar. Appl. Math., Vol.5, No.1, 1947

**** The semi-inverse method of constructing the stress field for a shell of arbitrary configuration was given by A.Love. Cf. A.Love, Mathematical Theory of Elasticity. ONTI, 1935, p.560

brium of the theory of elasticity, and boundary conditions of special form on the boundary surfaces applying the Castigliano principle, satisfied integrally Saint-Venant's compatibility conditions, and derived the natural boundary conditions on the contour surface. Thus, here the semi-inverse method led to a rather complete and convincing analysis of the question.

The semi-inverse method of solving the reduction problem is also found in the work of S.A.Ambartsumyan (Bibl.16a-c), and A.A.Khachatryan (Bibl.33).

In these studies, the components σ^{i3} (i = 1, 2) of the stress tensor were first expressed by the product of a certain prescribed function f(z) and the function $\varphi^i(x^j)$(j = 1, 2) which had to be determined. The component σ^{33} was taken as zero. The function f(z) was most often expressed by the equation

$$f(z) = \frac{1}{2}(z^2 - h^2). \tag{a}$$

More general forms of the function f(z) were also considered.

The expressions for f(z) similar to (a), as well as the condition that σ^{33} shall vanish, do not permit satisfaction of the boundary conditions on the boundary surfaces, except for the case when there is no load on them.

The displacement vector components u^i (i = 1, 2) were determined from the expressions for the components σ^{i3} on the basis of Hooke's law in terms of the functions f(z), $\varphi^i(x^j)$, and the derivatives of u^3 with respect to the coordinates x^j. This solved the problem of reduction, and the further formulation of the problem proceeded in the usual context of shell theory. If we turn to our approximation equations (7.5a) - (7.5b), it will be noted that, including terms with the factor $\frac{1}{2}(z^2 - h^2)$, these equations also contain additional

terms depending on the load on the boundary surfaces and permitting satisfaction of the boundary conditions on them.

The relations (7.5a) - (7.5b) confirm the applicability of the expression of the function f(z) by the equation (a) in the absence of loads on the boundary surfaces of the shell. Even in this case, however, the components σ^{33} cannot be equated to zero.

3. Reduction by Determining the Coefficients of the Expansion of the Displacement Vector Components in Series, in Special Functions of the z Coordinate /172

I.N.Vekua (Bibl.18) formulated the boundary problems of the theory of shells of variable thickness, solvable by calculating the coefficients of the expansion of the elastic displacement components in series in Legendre polynomials of the coordinates $x^3 = z$. The content of Sect.24 of the present book also belongs to this trend.

In Sect.24 we gave a method of determining the coefficients of the Fourier

expansions of displacement vector components in trigonometric series over the segment (-h, +h) of a normal to the basic surface of the shell. Here we used the general equation of dynamics (15.1). Obviously, this general equation permits the construction of equations for determining the coefficients of the expansion of the required quantities in series in any special function, and to find the general formulations of the corresponding boundary problems.

4. Generalized Formulations of the Dynamic Problems of the Theory of Plates and Shells

In the last decade, a new direction has developed in the dynamics of plates and shells, with a characteristic departure from the classical formulation of the corresponding boundary problem and the use of refined equations. The problems that encouraged the development of this line of investigation were mentioned in Subsection 1 of this Section.

We shall not analyze the numerous investigations by Soviet and foreign authors in this field of applied theory of elasticity. These studies were characterized by the desire to obtain an approximate mathematical description of certain restricted classes of dynamic processes in shells, which with sufficient accuracy reflect the experimental facts and the conclusions from certain exact solutions of three-dimensional dynamic problems.

As an example of the studies belonging to this trend, we might cite other authors (Bibl.29,32) who used the method given by us (Bibl.23a,b) and extended it to the dynamic problem of the theory of plates and cylindrical shells. The general theory of shells was not touched in these studies*. The work of (Bibl.32) derives an approximate theory of wave processes in plates and /173
shells which satisfactorily represents the experimental results and the conclusions from the solutions of three-dimensional problems.

Section 31. Comparison of Various Methods of Reduction

In conclusion, we shall give a brief comparison of the various methods of reduction, demanding again optimum satisfaction of the equations of the mathematical theory of elasticity by the solutions found from the equations of shell theory.

It is well known that exact solutions of the boundary problems of elasticity theory must satisfy the equations of motion, the Saint-Venant compatibility conditions, the relations resulting from Hooke's law, and the boundary

* With respect to the work of (Bibl.29) we must make two statements.
a) It is impossible to construct an approximate theory "free from hypotheses", since any method of formulating an approximate theory will contain some a priori postulate, for example the postulate that it is possible to base the theory on a finite segment of a Taylor series expressing the function that is to be determined. Thus, we can speak only of the relative accuracy of an approximate theory advanced.
b) The author refers to the work by Kennard and to another monograph (Bibl.3b), ignoring earlier investigations, although the method given there is directly connected with the content of the author's own work.

and initial conditions. Let us now consider the approximate systems of equations of the dynamics of shells proposed by us.

1. Equations Obtained by Use of Expansion in Tensor Series

The basis of discussion here is the series expansion of the displacement vector components. In terms of the coefficients of these expansions we express the coefficients of the expansions in series of the strain and stress tensor components. For this reason we satisfy: a) the Saint-Venant compatibility equations; b) Hooke's law; c) the conditions on the boundary surfaces of the shell.

The equations of motions are satisfied approximately, since we used them in the first version only to determine the coefficients of the expansion in tensor series of the displacement vector components, and subsequently used only finite segments of these series. In the second version, which is close to the classical theory, the equations of motion are used in the integral form.

2. Equations Resulting from the D'Alembert-Lagrange Principle

By analogy to the preceding, here we satisfy the Saint-Venant compatibility conditions and Hooke's law. The conditions on the boundary surface are included in the equations of motion. The equations of motion are satisfied integrally and approximately in consequence of the restriction of the number of degrees of freedom of the shell in the direction of the coordinates $x^3 = z$.

For comparison, we recall that the solutions of the equations of the statics of plates given by E.Reissner satisfy the equations of equilibrium of the theory of elasticity, Hooke's law, the boundary conditions on the boundary surfaces of the plate, and integrally (approximately) the compatibility conditions (Bibl.20a). /174

Consequently, each method of reduction permits us to find only an approximate solution of the three-dimensional problem of the theory of elasticity, with the character of the error depending substantially on the reduction method. There is no method of reduction that does not involve some assumptions expressed in geometrical or analytic form.

CHAPTER IV /175

APPROXIMATELY EQUIVALENT SYSTEMS

Section 1. Introductory Remarks

The question of reducing the three-dimensional problem of the theory of elasticity to a two-dimensional problem of the theory of shells is a special case of the more general problem of the approximate replacement of one material system by another which is close to the first one by some definite criterion.

For example, in reducing the three-dimensional problem of the theory of elasticity to a two-dimensional problem, we replaced the three-dimensional elastic continuum-shell by a certain medium, having the properties of continuity in two-dimensional space and a finite number of degrees of freedom in the third dimension. Thus the shell, in V.Z.Vlasov's terminology, is a discrete-continuum system (Bibl. 3b). This system approximately replaces the three-dimensional elastic body. Clearly, the problem of constructing approximately equivalent systems is considerably broader than the problem of reduction.

The construction of approximately equivalent material systems is related in its meaning to the determination of a system of functions approximately representing another prescribed system of functions. For this reason it is natural to use the analytic apparatus of the theory of approximation in solving the problem of constructing approximately equivalent systems.

In this Chapter we shall consider the application of certain methods of the theory of approximation functions to the finding of approximate analytic statements of the dynamic boundary problems of the theory of shells.

Section 2. First Method of Linear Approximation of the Components of the Stress Tensor and the Finite-Deformation Tensor

1. On the Construction of an Isotropic, Approximately Equivalent, Elastic Body

In the theory of small deformations it is assumed that the nonlinear terms entering into the composition of the components of the finite-deformation tensor (II, 2.11) can be neglected, without introducing a substantial error /176
into the field of stresses.

Here we shall consider the linear approximation of the components of the finite-deformation tensor by the components of the small-deformation tensor. Such an approximation is a consequence of the construction of an elastic medium approximately representing the motion of the corresponding elements of the body considered in the initial formulation of some nonlinear problem of elastodynamics, by the motion of its elements. As a result we obtain a better field of stresses than in the theory of small deformations.

Let us imagine that two elastic bodies with non-coinciding elastic constants have, in the undeformed state, the same geometrical form and dimensions and are referred to identical systems of Lagrangian coordinates x^i. Let the deformed state of the first body be characterized by the finite-deformation tensor D_{ik}, and the deformed state of the second body by the small-deformation tensor ε_{ik}, which enters into the linear part of the components of the tensor D_{ik}.

We shall consider these bodies as approximately equivalent material systems if the elastic constants of the second body are such that they satisfy the condition of the least-square deviation of their specific potential energies of deformation in some region Ω of variation of the tensor components Φ_{ki}, defined elsewhere (II, 2.5). We recall that the tensor components D_{ik} and ε_{ik} are constructed from the tensor components Φ_{ki}, as we have shown in Chapter II.

The points of the region Ω are individualized by the coordinates

$$x_{ik} = \Phi_{ik}. \tag{2.1}$$

Consequently, the region Ω is a nine-dimensional space. Let us assume that each of the coordinates x_{ik} varies from zero to some positive and negative quantity a_{ik}, which may be selected, for example, on the basis of the requirement that the first body shall have no plastic deformations. It is also possible to use other methods for selecting the quantities a_{ik}, based on kinematic considerations connected with the restrictions imposed on the components of Ω_{ik} determined previously (II, 2.7). We shall make use of the kinematic restrictions in determining the boundaries of the region Ω in the following subsection, but here we will use the condition of the absence of plastic deformations.

Let us assume first that the boundary of the region Ω is known. Making use of (II, 11.2a), and assuming that both bodies are isotropic, we find the specific potential energy of deformation of the first body:

$$2\Pi = \lambda (g^{rs} D_{rs})^2 + 2\mu g^{ir} g^{ks} D_{ik} D_{rs}. \tag{2.2a}$$

The specific potential energy of the second body is expressed similarly: /177

$$2\Pi^* = \lambda^* (g^{rs} \varepsilon_{rs})^2 + 2\mu^* g^{ir} g^{ks} \varepsilon_{ik} \varepsilon_{rs}. \tag{2.2b}$$

Making use of eqs.(II, 2.5), (II, 2.6), (II, 2.11) and (2.1), we find

$$\varepsilon_{rs} = \frac{1}{2} (x_{rs} + x_{sr}); \tag{2.3a}$$

$$D_{rs}=\varepsilon_{rs}+\frac{1}{2}g^{jp}x_{rp}x_{sj} \tag{2.3b}$$

and after elementary transformations we obtain

$$2\Pi=\lambda(g^{rs}\varepsilon_{rs})^2+2\mu g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs}+\lambda\Big[(g^{rs}\varepsilon_{rs})(g^{rs}g^{jp}x_{rp}x_{sj})+$$
$$+\frac{1}{4}(g^{rs}g^{jp}x_{rp}x_{sj})^2\Big]+2\mu\Big[g^{ir}g^{ks}g^{jp}\varepsilon_{ik}x_{rp}x_{sj}+$$
$$+\frac{1}{4}g^{ir}g^{ks}g^{qm}g^{jp}x_{im}x_{kq}x_{rp}x_{sj}\Big]. \tag{2.4}$$

Consider the integral

$$I=\int_{(\Omega)}[\Pi^*-\Pi]^2\,d\Omega=\int_{(\Omega)}\Big\{\frac{1}{2}(\lambda^*-\lambda)(g^{rs}\varepsilon_{rs})^2+(\mu^*-\mu)\,g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs}-$$
$$-\frac{1}{2}\lambda\Big[(g^{rs}\varepsilon_{rs})(g^{rs}g^{jp}x_{rp}x_{sj})+\frac{1}{4}(g^{rs}g^{jp}x_{rp}x_{sj})^2\Big]-$$
$$-\mu\Big[g^{ir}g^{ks}g^{jp}\varepsilon_{ik}x_{rp}x_{sj}+\frac{1}{4}g^{ir}g^{ks}g^{qm}g^{jp}x_{im}x_{kq}x_{rp}x_{sj}\Big]\Big\}^2\,d\Omega. \ (2.5) \tag{2.5}$$

Let us find the elastic components λ^* and μ^* of the second body from the condition that I shall be minimum. Equating the derivatives $\frac{\partial I}{\partial\lambda^*}$ and $\frac{\partial I}{\partial\mu^*}$, to zero, we obtain the following system of linear algebraic equations

$$\begin{aligned} b_{11}(\lambda^*-\lambda)+b_{12}(\mu^*-\mu)&=b_1,\\ b_{21}(\lambda^*-\lambda)+b_{22}(\mu^*-\mu)&=b_2. \end{aligned} \tag{2.6}$$

where

$$b_{11}=\frac{1}{2}\int_{(\Omega)}(g^{rs}\varepsilon_{rs})^4\,d\Omega;\quad b_{12}=\int_{(\Omega)}(g^{rs}\varepsilon_{rs})^2\,g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs}d\Omega; \tag{2.7a}$$

$$b_{21}=\int_{(\Omega)}(g^{rs}\varepsilon_{rs})^2\,g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs}d\Omega=b_{12};\quad b_{22}=2\int_{(\Omega)}(g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs})^2\,d\Omega; \tag{2.7b}$$

$$b_1=\frac{1}{2}\int_{(\Omega)}(g^{rs}\varepsilon_{rs})^2\Big\{\lambda\Big[(g^{rs}\varepsilon_{rs})(g^{rs}g^{jp}x_{rp}x_{sj})+\frac{1}{4}(g^{rs}g^{ip}x_{rp}x_{si})^2\Big]+ \tag{2.7c}$$

$$+ 2\mu \left[g^{ir} g^{ks} g^{jp} \varepsilon_{ik} x_{rp} x_{sj} + \frac{1}{4} g^{ir} g^{ks} g^{qm} g^{jp} x_{im} x_{kq} x_{rp} x_{sj} \right] \Bigg\} d\Omega;$$ /178

(2.7c)

$$b_2 = \int_{(\Omega)} g^{ir} g^{ks} \varepsilon_{ik} \varepsilon_{rs} \Bigg\{ \lambda \left[(g^{rs} \varepsilon_{rs}) (g^{rs} g^{jp} x_{rp} x_{sj}) + \frac{1}{4} (g^{rs} g^{jp} x_{rp} x_{sj})^2 \right] +$$

$$+ 2\mu \left[g^{ir} g^{ks} g^{jp} \varepsilon_{ik} x_{rp} x_{sj} + \frac{1}{4} g^{ir} g^{ks} g^{qm} g^{jp} x_{im} x_{kq} x_{rp} x_{sj} \right] \Bigg\} d\Omega.$$

(2.7d)

It can be shown that the determinant of the system of equations (2.6) is nonzero. We can convince ourselves of this by determining the elastic constants λ^* and μ^* from the system (2.6).

Since the linearization performed here relates to the stressed-strained state of one of the elements of an elastic body, i.e., since it is local, let us introduce a local Cartesian rectilinear system of coordinates connected with this element. In this system, the components of the metric tensor are expressed by the well-known equations: $g_{ii} = 1$; $g_{ik} = 0$; $(i \neq k)$.

To simplify the solution of this problem without conflicting with its physical content, let us substitute for the region of integration Ω the extended region Ω^*, assuming that each of the nine coordinates x_{ik} varies from $-a$ to $+a$, where a is the greatest of the absolute values assumed by the coordinates x_{ik} on the boundary of the region Ω. Under this condition, the region Ω will be included in Ω^*. Let us find, under these assumptions, the expressions for the coefficients b_{ik} and b_j determined by eqs.(2.7a) - (2.7b).

From eq.(2.3a) we obtain

$$g^{rs} \varepsilon_{rs} \overset{*}{=} \sum_{i=1}^{3} x_{ii}; \quad g^{ir} g^{ks} \varepsilon_{ik} \varepsilon_{rs} \overset{*}{=} \frac{1}{4} \sum_{i=1}^{3} \sum_{k=1}^{3} (x_{ik} + x_{ki})^2; \tag{2.8a}$$

$$g^{rs} g^{jp} x_{rp} x_{sj} \overset{*}{=} \sum_{r=1}^{3} \sum_{j=1}^{3} x_{rj}^2; \tag{2.8b}$$

$$g^{ir} g^{ks} g^{jp} \varepsilon_{ik} x_{rp} x_{sj} \overset{*}{=} \frac{1}{2} \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} (x_{ik} + x_{ki}) x_{ij} x_{kj}; \tag{2.8c}$$

$$g^{ir} g^{ks} g^{qm} g^{jp} x_{im} x_{kq} x_{rp} x_{sj} \overset{*}{=} \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{q=1}^{3} x_{ij} x_{kj} x_{iq} x_{kq}. \tag{2.8d}$$

We put

$$x_{ik}=a\xi_{ik} \qquad (i,\ k=1,\ 2,\ 3);$$

Consequently, /179

$$b_{11}=\frac{a^{13}}{2}\int_{-1}^{1}\cdots\int_{-1}^{1}\left(\sum_{i=1}^{3}\xi_{ii}\right)^{4}d\xi_{11}d\xi_{22}\ldots d\xi_{32}; \tag{2.9a}$$

$$b_{12}=b_{21}=\frac{a^{13}}{4}\int_{-1}^{1}\cdots\int_{-1}^{1}\left(\sum_{i=1}^{3}\xi_{ii}\right)^{2}\sum_{i=1}^{3}\sum_{k=1}^{3}(\xi_{ik}+\xi_{ki})^{2}d\xi_{11}\ldots d\xi_{32}; \tag{2.9b}$$

$$b_{22}=\frac{a^{13}}{8}\int_{-1}^{1}\cdots\int_{-1}^{1}\left[\sum_{i=1}^{3}\sum_{k=1}^{3}(\xi_{ik}+\xi_{ki})^{2}\right]^{2}d\xi_{11}d\xi_{22}\ldots d\xi_{32}. \tag{2.9c}$$

In calculating b_i, all terms containing odd powers of the variables ξ_{ik} must be excluded in advance, since the limits of integration are symmetric. We find

$$b_{1}=\frac{a^{15}}{8}\int_{-1}^{1}\cdots\int_{-1}^{1}\left(\sum_{i=1}^{3}\xi_{ii}\right)^{2}\left\{\lambda\left(\sum_{r=1}^{3}\sum_{j=1}^{3}\xi_{rj}^{2}\right)^{2}+\right.$$
$$\left.+2\mu\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3}(\xi_{ik}\xi_{kj})^{2}\right\}d\xi_{11}d\xi_{22}\ldots d\xi_{32}; \tag{2.10a}$$

$$b_{2}=\frac{a^{15}}{16}\int_{-1}^{1}\cdots\int_{-1}^{1}\sum_{i=1}^{3}\sum_{k=1}^{3}(\xi_{ik}+\xi_{ki})^{2}\left\{\lambda\left(\sum_{r=1}^{3}\sum_{j=1}^{3}\xi_{rj}^{2}\right)^{2}+\right.$$
$$\left.+2\mu\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3}(\xi_{ij}\xi_{kj})^{2}\right\}d\xi_{11}d\xi_{22}\ldots d\xi_{32}. \tag{2.10b}$$

To calculate the nine-fold integrals entering into the expression for the coefficients of eqs.(2.6), it is appropriate to use approximation formulas. The analytic properties of the integral expressions are very simple and permit the use of formulas approximately expressing double and triple integrals*.

* Cf., for instance, Sh. Ye. Mikeladze, Numerical Methods of Mathematical Analysis, Gostekhizdat, 1953, p.507.

Performing the calculations, we find

$$b_{11} \cong 665,7a^{13};\ b_{12} = b_{21} \cong 1161,0a^{13};\ b_{22} \cong 4847,0a^{13}; \qquad (2.11a)$$

$$b_1 \cong (732,9\lambda + 561,2\mu)\, a^{15};\ b_2 \cong (2890,0\lambda + 2245,0\mu)\, a^{15}. \qquad (2.11b)$$

From the system of equations (2.6) we obtain

$$\lambda^* = \lambda(1 + \alpha_{11}a^2) + \mu\alpha_{12}a^2; \qquad (2.12a)$$

$$\mu^* = \lambda\alpha_{21}a^2 + \mu(1 + \alpha_{22}a^2). \qquad (2.12b)$$

where /180

$$\alpha_{11} \cong 0,1057,\quad \alpha_{12} \cong 0,0611,$$
$$\alpha_{21} \cong 0,5709,\quad \alpha_{22} \cong 0,4485. \qquad (2.13)$$

The terms in a^2 approximately determine the effect exerted by the nonlinear terms, entering into the components of the finite-deformation tensor, on the stress-tensor components. These terms are equivalent to a certain increase in the Lamé constants.

We note that in the case of the variation of x_{ik} over nonsymmetric intervals, terms linear in a would enter into eqs.(2.12a) - (2.12b).

Now, on the basis of (II, 4.5b), we can write the following relations:

$$\sigma_{ik} = \lambda g_{ik}g^{rs}D_{rs} + 2\mu D_{ik} \cong \lambda^* g_{ik}g^{rs}\varepsilon_{rs} + 2\mu^*\varepsilon_{ik} \qquad (2.14)$$
$$(i,\ k,\ r,\ s = 1,\ 2,\ 3).$$

Hence, we find

$$D_{ik} = \frac{1}{2\mu}\left[\left(\lambda^* - \frac{3\lambda^* + 2\mu^*}{3\lambda + 2\mu}\lambda\right) g_{ik}g^{rs}\varepsilon_{rs} + 2\mu^*\varepsilon_{ik}\right], \qquad (2.15a)$$

or

$$D_{ik} = \alpha g_{ik}g^{rs}\varepsilon_{rs} + \beta\varepsilon_{ik}. \qquad (2.15b)$$

where

$$\alpha = \frac{1}{2\mu}\left(\lambda^* - \frac{3\lambda^* + 2\mu^*}{3\lambda + 2\mu}\lambda\right);\quad \beta = \frac{\mu^*}{\mu}. \qquad (2.15c)$$

Equations (2.1.4), (2.15a) - (2.15c) determine the required linear approximations of the stress-tensor components and of the finite-deformation tensor.

Several remarks will be given on the results. In the construction of an elastic body, approximately equivalent to a body with finite deformations, we assumed that the body to be constructed was isotropic. If we abandon this assumption it would be possible to dispose of a larger number of elastic constants and decrease the mean-square deviation of the specific potential energy Π from the potential energy Π^* in the region Ω. Consequently, the construction of a body approximately equivalent in the energetic criterion to a body with finite deformations will lead to the consideration of an anisotropic elastic medium.

2. Connection with the Theory of Optimum Systems

The method given above for the construction of an elastic body which is energetically approximately equivalent, is closely related in meaning to the construction of what is called an optimum system, which is known from the theory of automatic control*.

In certain problems connncted with the theory of noise, a linear function is separated from the random function describing a dynamic process including white noise. This separation is based on the condition of the minimum of the corresponding mean-square error. /181

This problem is analogous to the above problem, which leads to a separation of the linear functions of the tensor components Φ_{ik} from the components of the finite-deformation tensor. Here we have in mind not only an external similarity, but a more profound analogy of problems whose physical content is different. Indeed, the process of variation of the stressed-strained state of an elastic body has under actual conditions a random character and belongs in the field of problems studied by probability methods (Bibl.2b). The separation of the linear part of the stress and strain tensor components, based on the requirement of a minimum of the corresponding mean-square deviation of the potential energies Π and Π^* may, in this connection, be regarded as a practically justified simplification of the mathematical description of a complex phenomenon, permitting a separation of the "principal parts" of the quantities under study.

It is easy to establish a direct correlation between the above-described method for the construction of a system approximately equivalent as to the energetic criterion, and the methods of probability. This, however, would go beyond the scope of the present investigation.

3. Determination of the Parameter a

As already stated, the parameter a can be determined from various physical requirements imposed on the components of the tensors D_{ik} and Ω_{ik}.

We shall start out from the Huber-Mises plasticity condition. According to this condition and to the connection between the intensities of stresses

* Cf. V.S. Pugachev, Theory of Random Functions and its Application to Problems of Automatic Control, Chapter 16. Fizmatgiz, 1960

and those of the strains, let the region of elastic deformations of the material be determined by the condition imposed on the intensity of the deformations. This condition, in the Cartesian system of rectangular coordinates, has the following form:*

$$\sqrt{(D_{11}-D_{22})^2+(D_{22}-D_{33})^2+(D_{33}-D_{11})^2+6(D_{12}^2+D_{13}^2+D_{23}^2)} \leq A, \tag{2.16}$$

where A is a certain physical constant**.

To determine the boundaries of the region Ω^*, let us set all the coordinates x_{ik}, except one, as equal to zero and then let us find the values of the nonzero coordinate on the boundaries of the elasticity region from eq.(2.16). From the resultant values of $|x_{ik}|_\Omega$, let us select the greatest and assume that a is equal to this quantity. Let /182

$$x_{11} \neq 0;\ x_{12} = \ldots = x_{32} = 0; \tag{a}$$

Then,

$$D_{11} = x_{11} + \frac{1}{2} x_{11}^2;\quad D_{22} = \ldots = D_{23} = 0; \tag{b}$$

Consequently,

$$|x_{11}|_\Omega^2 + 2|x_{11}|_\Omega - A\sqrt{2} = 0;\ |x_{11}|_\Omega = \sqrt{1 + A\sqrt{2}} - 1. \tag{c}$$

Let us put further

$$x_{12} \neq 0;\ x_{11} = x_{22} = \ldots = x_{32} = 0; \tag{d}$$

Then,

$$D_{11} = \frac{1}{2} x_{12}^2;\quad D_{12} = \frac{1}{2} x_{12};\quad D_{12} = \ldots = D_{32} = 0, \tag{e}$$

From eq.(2.16), we find

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* Cf. A.A.Il'yushin, Plasticity, Gostekhizdat, 1948; L.M.Kachanov, Foundations of the Theory of Plasticity, Gostekhizdat, 1956.

** The introduction of the components of the finite-deformation tensor into the condition (2.16) is controversial, since the condition (2.16) belongs to the theory of small elasto-plastic deformations, which, in particular, is noted in the book by R.Hill "Mathematical Theory of Plasticity". The transition in the conditions (2.16) to the components of the small-deformation tensor introduces no substantial changes in the conclusions of this Subsection.

$$|x_{12}|_2^4+3|x_{12}|_2^2-2A^2=0;\quad |x_{12}|_2=\sqrt{\frac{\sqrt{9+8A^2}-3}{2}}. \qquad (f)$$

Comparing eqs.(c) and (f) we find that a is expressed by the equations

$$a=|x_{12}|_2=\sqrt{\frac{3}{2}}\sqrt{\sqrt{1+\frac{8}{9}A^2}-1}. \qquad (2.17a)$$

For sufficiently small values of A we may approximately put

$$a\cong\sqrt{\frac{2}{3}}A\cong 0{,}8A. \qquad (2.17b)$$

Of course, this method of determining the region Ω^* is not perfect. Essentially we have confined ourselves to the results of "sounding" the region Ω only in the direction of the axes of the multi-dimensional coordinate system x_{ik} and, in addition, we have used highly simplified concepts as to the structure of the region Ω. The method of "sounding" used here does not reflect the influence of the components of the antisymmetric tensor Ω_{ik} on the nonlinear terms contained in the components of the finite-deformation tensor D_{ik}. We have likewise considered that the positive and negative signs for the coordinates x_{ik} were equally probable. This led us, in particular to the conclusion that $\lambda^* > \lambda$ and $\mu^* > \mu$.

These conclusions may be illustrated by an elementary one-dimensional /183 example. From the relation

$$\sigma_{11}=E\left(\epsilon_{11}+\frac{1}{2}\epsilon_{11}^2\right)=E^*\epsilon_{11} \qquad (g)$$

it follows that, for $\epsilon_{11}>0$, $E^*>E$, while for $\epsilon_{11}<0$, $E^*<E$, and that the absolute value of the difference E^*-E is the same in these cases. However, in approximation on the symmetric interval of the potential energy $\frac{1}{2}E(\epsilon_{11}+\frac{1}{2}\epsilon_{11}^2)^2$ by the energy $\frac{1}{2}E^*\epsilon_{11}^2$, we get the result that not always $E^*>E$.

The one-dimensional case differs from the others precisely in that all the components of Ω_{ik} vanish here. This confirms the conclusion that our deductions are insufficient, because of the fact that they are based on a simplified concept of the structure of the region Ω and on the use of approximation over the symmetric interval.

These shortcomings may prove substantial for the case of shells, since considerable displacements and rotations of the elements may take place there, without the appearance of plastic zones. Let us, therefore, consider a different choice of independent variables and replace the region of approximate representation of the specific potential energy of finite deformations Π by

the specific potential energy of small deformations Π^*.

Section 3. Second Method of Linear Approximation of the Components of the Stress Tensor and of the Finite-Deformation Tensor

Let us return to eq.(II, 9.1):

$$D_{ik} = \varepsilon_{ik} + \frac{1}{2} g^{rj} (\varepsilon_{ir}\varepsilon_{kj} + \varepsilon_{ir}\Omega_{kj} + \varepsilon_{kj}\Omega_{ir} + \Omega_{ir}\Omega_{kj}). \qquad \text{(a)}$$

This equation shows that it is possible to utilize the nine quantities ε_{ik} and Ω_{ik} directly as coordinates of the region Ω. There is no need, however, for repeating all the calculations given in the last Section.

Let us first fix the components Ω_{ik}, considering them as certain parameters. This is equivalent to a separation, in the nine-dimensional space Ω, of a six-dimensional space of deformations ω.

To calculate the integrals on the right-hand sides of eqs.(2.6), it is sufficient to consider the transformation of coordinates according to the formulas

$$x_{ik} = \varepsilon_{ik} + \Omega_{ik}. \qquad (3.1)$$

Let

$$a = \max |\varepsilon_{ik}|_{\Omega}, \qquad (3.2)$$

where $\max |\varepsilon_{ik}|_{\Omega}$ is the greatest absolute magnitude, among the set of values, taken by the component ε_{ik} on the boundary of the region ω. The quantity a is determined from the condition (2.16). Hereafter, in calculating the integrals entering into the expressions λ^* and μ^*, we shall consider the extended region ω^* (Sect.2) determined by the quantity a. /184

Let us now put

$$\varepsilon_{ik} = a\eta_{ik}. \qquad (3.3)$$

The variables η_{ik} may vary over arbitrary intervals lying within the range (-1, +1).

Bearing in mind the conclusions of our study of approximation over a symmetric range, drawn in the last Section, let us assume that all the quantities η_{ik} vary over an interval (α, β) where $|\alpha|$ and $|\beta|$ are proper fractions. The choice of the total range of variation for all the variables η_{ik} is a substantial simplification of the problem. In performing specific calculations we shall most often assume that $\alpha = 0$, $\beta = 1$ or $\alpha = -1$, $\beta = 0$. However, most of the conclusions drawn below do not depend on the values of α and β. Let us

return to eqs.(2.7a) - (2.7d) and (2.8a) - (2.8d) in order to find new expressions for the coefficients b_{ik} and b_i of eqs.(2.6). Bearing in mind eqs.(3.3), we obtain instead of eq.(2.8a)

$$g^{rs}\varepsilon_{rs} \overset{*}{=} a\sum_{i=1}^{3}\eta_{ii}; \quad g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs} \overset{*}{=} a^2\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{ik}^2. \tag{3.4}$$

Equations (2.7a) - (2.7b) now take the following form:

$$b_{11} = \frac{1}{2}a^{10}\int_\alpha^\beta\ldots\int_\alpha^\beta\left(\sum_{i=1}^{3}\eta_{ii}\right)^4 d\eta_{11}\ldots d\eta_{23}, \tag{3.5a}$$

$$b_{12} = b_{21} = a^{10}\int_\alpha^\beta\ldots\int_\alpha^\beta\left(\sum_{i=1}^{3}\eta_{ii}\right)^2\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{ik}^2\right) d\eta_{11}\ldots d\eta_{23}, \tag{3.5b}$$

$$b_{22} = 2a^{10}\int_\alpha^\beta\ldots\int_\alpha^\beta\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{ik}^2\right)^2 d\eta_{11}\ldots d\eta_{23}. \tag{3.5c}$$

In calculating the quantity b_i let us make use of the variables x_{ik}. Applying eqs.(3.1), (3.3) and making use of eqs.(2.7c) - (2.7d), we find

$$b_1 = \frac{1}{2}a^8\int_\alpha^\beta\ldots\int_\alpha^\beta\left(\sum_{i=1}^{3}\eta_{ii}\right)^2(\lambda\Phi_1 + 2\mu\Phi_2)\, d\eta_{11}\ldots d\eta_{23}, \tag{3.6a}$$

$$b_2 = a^8\int_\alpha^\beta\ldots\int_\alpha^\beta\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{ik}^2\right)(\lambda\Phi_1 + 2\mu\Phi_2)\, d\eta_{11}\ldots d\eta_{23}. \tag{3.6b}$$

From eqs.(2.7c) - (2.7d), (2.8a) - (2.8b), and (3.3) we obtain /185

$$\Phi_1 = (g^{rs}\varepsilon_{rs})(g^{rs}g^{jp}x_{rp}x_{sj}) + \frac{1}{4}(g^{rs}g^{jp}x_{rp}x_{sj})^2 =$$

$$= a\left(\sum_{i=1}^{3}\eta_{ii}\right)\left(a^2\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2 + A^2\right) + \frac{1}{4}\left(a^2\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2 + A^2\right)^2. \tag{3.7a}$$

where

$$A^2 = \sum_{j=1}^{3}\sum_{k=1}^{3}\Omega_{jk}^2; \tag{3.7b}$$

$$\Phi_2 = g^{ir} g^{ks} g^{jp} \varepsilon_{ik} x_{rp} x_{sj} + \frac{1}{4} g^{ir} g^{ks} g^{qm} g^{jp} x_{im} x_{kq} x_{rp} x_{sj}. \tag{b}$$

Calculating the summands in the right-hand side of eq.(b), we find, on the basis of eqs.(3.1), after several transformations and after proofs that the sums linearly containing Ω_{ik} vanish*, the following

$$g^{ir} g^{ks} g^{jp} \varepsilon_{ik} x_{rp} x_{sj} = \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} \varepsilon_{ik} x_{ij} x_{kj} = \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} \varepsilon_{ik} \varepsilon_{ij} \varepsilon_{kj} + \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1} \varepsilon_{ik} \Omega_{ij} \Omega_{kj}^{1}. \tag{3.7c}$$

We have, further,

$$\begin{aligned} g^{ir} g^{ks} g^{qm} g^{jp} x_{im} x_{kq} x_{rp} x_{sj} &= \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{q=1}^{3} x_{ij} x_{kj} x_{iq} x_{kq} = \\ &= \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{q=1}^{3} \varepsilon_{ij} \varepsilon_{kj} \varepsilon_{iq} \varepsilon_{kq} + 2 \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{q=1}^{3} \varepsilon_{ij} \varepsilon_{kj} \Omega_{iq} \Omega_{kq} + \\ &+ 2 \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{q=1}^{3} \varepsilon_{ij} \varepsilon_{iq} \Omega_{kj} \Omega_{kq} + 2 \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{q=1}^{3} \varepsilon_{ij} \varepsilon_{qk} \Omega_{kj} \Omega_{iq} + \\ &= \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{q=1}^{3} \Omega_{ij} \Omega_{kj} \Omega_{iq} \Omega_{kq}. \end{aligned} \tag{3.7d}$$

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* The triple sums entering into eq.(3.7c) may be calculated, for example, on the basis of the following equation:

$$\begin{aligned} \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{j=1}^{3} \varepsilon_{ik} x_{ij} x_{kj} &= \varepsilon_{11} (x_{11}^2 + x_{12}^2 + x_{13}^2) + \varepsilon_{22} (x_{21}^2 + x_{22}^2 + x_{23}^2) + \\ &+ \varepsilon_{33} (x_{31}^2 + x_{32}^2 + x_{33}^2) + 2\varepsilon_{12} (x_{11} x_{21} + x_{12} x_{22} + x_{13} x_{23}) + \\ &+ 2\varepsilon_{13} (x_{11} x_{31} + x_{12} x_{32} + x_{13} x_{33}) + 2\varepsilon_{23} (x_{21} x_{31} + x_{22} x_{32} + x_{23} x_{33}). \end{aligned}$$

Here, of course, one must remember that the quantities ε_{ik} are symmetric in the indices and that the Ω_{ik} are antisymmetric.

From the transformations leading to eq.(3.7d) it is clear that the sum of terms with an odd dimension relative to the components of the tensor Ω_{ik} must vanish. /186

Let us introduce the notation

$$\sum_{q=1}^{3} \Omega_{iq}\Omega_{kq} = 4B_{ik} = 4B_{ki}, \tag{3.8a}$$

$$\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{q=1}^{3} \Omega_{ij}\Omega_{kj}\Omega_{iq}\Omega_{kq} = \sum_{i=1}^{3}\sum_{k=1}^{3}\left(\sum_{j=1}^{3}\Omega_{ij}\Omega_{kj}\right)\times$$

$$\times\left(\sum_{q=1}^{3}\Omega_{iq}\Omega_{kq}\right) = \sum_{i=1}^{3}\sum_{k=1}^{3} B_{ik}^2 = 4C^2. \tag{3.8b}$$

Now, bearing in mind eqs.(3.3), we find

$$\Phi_2 = C^2 + a\sum_{i=1}^{3}\sum_{k=1}^{3} B_{ik}\eta_{ik} + 6a^2\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3} B_{ik}\eta_{ij}\eta_{kj} +$$

$$+ a^3\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3}\eta_{ik}\eta_{ij}\eta_{kj} + \frac{1}{4}a^4\sum_{i=1}^{3}\sum_{k=1}^{3}\left(\sum_{j=1}^{3}\eta_{ij}\eta_{kj}\right)^2. \tag{3.9}$$

The right-hand side of eq.(3.9) can be represented in a somewhat different form. Let us introduce the notation

$$\sum_{j=1}^{3}\eta_{ij}\eta_{kj} = \xi_{ik} = \xi_{ki}. \tag{3.10}$$

Then, /187

$$\Phi_2 = C^2 + a\sum_{i=1}^{3}\sum_{k=1}^{3} B_{ik}\eta_{ik} + 6a^2\sum_{i=1}^{3}\sum_{k=1}^{3} B_{ik}\xi_{ik} +$$

$$+ a^2\sum_{i=1}^{3}\sum_{k=1}^{3}\xi_{ik}\eta_{ik} + \frac{1}{4}a^4\sum_{i=1}^{3}\sum_{k=1}^{3}\xi_{ik}^2. \tag{3.11}$$

From eqs.(3.6a) - (3.6b), (3.7a), (3.9) - (3.11) result the following general representations of the quantities b_i :

$$b_1=\frac{1}{2}a^8\Big\{\lambda\,(c_{10}A^4+c_{11}A^2a+c_{12}A^2a^2+c_{13}a^3+c_{14}a^4)+$$
$$+2\mu\Big[d_{10}C^2+a\sum_{i=1}^{3}\sum_{k=1}^{3}B_{ik}\,(\alpha_{ik}+a\beta_{ik})+d_{13}a^3+d_{14}a^4\Big]\Big\};\tag{3.12a}$$

$$b_2=a^8\Big\{\lambda\,(c_{20}A^4+c_{21}A^2a+c_{22}A^2a^2+c_{23}a^3+c_{24}a^4)+2\mu\Big[d_{20}C^2+$$
$$+a\sum_{i=1}^{3}\sum_{k=1}^{3}B_{ik}\,(\gamma_{ik}+a\delta_{ik})+d_{23}a^3+d_{24}a^4\Big]\Big\}.\tag{3.12b}$$

In these equations, the coefficients c_{ik}, d_{ik}, α_{ik}, β_{ik}, γ_{ik}, δ_{ik} do not depend on the parameter a nor the components of the tensor Ω_{ik}. These coefficients are expressed by the following sextuple integrals over the region ω, arithmetized by the coordinates η_{ik} :

$$c_{10}=\frac{1}{4}\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^2 d\omega;\quad c_{11}=\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^3 d\omega;$$

$$c_{12}=\frac{1}{2}\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^2\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)d\omega;\quad c_{13}=\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^3\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)d\omega;$$

$$c_{14}=\frac{1}{4}\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^2\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)^2 d\omega;\quad c_{20}=\frac{1}{4}\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{ik}^2\Big)d\omega;$$

$$c_{21}=\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)d\omega;\quad c_{22}=\frac{1}{2}\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)^2 d\omega;$$

$$c_{23}=\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)\Big(\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{ik}^2\Big)^2 d\omega;\quad c_{24}=\frac{1}{4}\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{ik}^2\Big)^3 d\omega;\tag{3.13a}$$

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$$d_{10}=4c_{10};\quad d_{13}=\int\limits_{(\omega)}\left(\sum_{i=1}^{3}\eta_{ii}\right)^2\left(\sum_{l=1}^{3}\sum_{k=1}^{3}\xi_{ik}\eta_{ik}\right)d\omega;$$

$$d_{11}=\frac{1}{4}\int\limits_{(\omega)}\left(\sum_{i=1}^{3}\eta_{ii}\right)^2\left(\sum_{i=1}^{3}\sum_{k=1}^{2}\xi_{ik}^2\right)d\omega;\quad d_{20}=4c_{20};$$

$$d_{23}=\int\limits_{(\omega)}\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{ik}^2\right)\left(\sum_{l=1}^{3}\sum_{k=1}^{3}\xi_{ik}\eta_{ik}\right)d\omega;$$

$$d_{24}=\frac{1}{4}\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{ik}^2\right)\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\xi_{ik}^2\right)d\omega;\tag{3.13b}$$

$$\alpha_{ik}=\int\limits_{(\omega)}\left(\sum_{j=1}^{3}\eta_{jj}\right)^2\eta_{ik}d\omega;\quad \beta_{ik}=6\int\limits_{(\omega)}\left(\sum_{j=1}^{3}\eta_{jj}^2\right)^2\xi_{ik}d\omega;\tag{3.13c}$$

$$\gamma_{ik}=\int\limits_{(\omega)}\left(\sum_{r=1}^{3}\sum_{s=1}^{3}\eta_{rs}^2\right)\eta_{ik}d\omega;\quad \delta_{ik}=6\int\limits_{(\omega)}\left(\sum_{r=1}^{3}\sum_{s=1}^{3}\eta_{rs}^2\right)\xi_{ik}d\omega.\tag{3.13d}$$

Under the above assumptions as to the variation interval of η_{ik}, we have

$$\int\limits_{(\omega)}\ldots d\omega=\int\limits_{\alpha}^{\beta}\ldots\int\limits_{\alpha}^{\beta}\ldots d\eta_{11}\ldots d\eta_{23}.\tag{3.14}$$

We still have to determine the quantity a. Let us turn again to the condition of plasticity (2.16), replacing in it, according to the theory of small elasto-plastic deformations, the components of the finite-deformation tensor by the components of the small-deformation tensor.

The boundary of the plasticity region is a surface of the second order in the six-dimensional space of quantities ϵ_{ik}. Let us find the points of intersection of this surface with the axes ϵ_{ik} and let us find $|\epsilon_{ik}|_{2}$. Then, according to eq.(3.2), we shall find a.

Let us put first $\epsilon_{11}\neq 0$; $\epsilon_{22} = \ldots = \epsilon_{23} = 0$. Then, from eq.(2.16), we obtain

$$|\epsilon_{11}|_2=A\frac{\sqrt{2}}{2}.\tag{c}$$

Putting $\epsilon_{12}\neq 0$ and $\epsilon_{11} = \ldots = \epsilon_{23} = 0$, we find

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$$\varepsilon_{12} = \frac{A}{\sqrt{6}}. \qquad (d)$$

Comparing eqs.(c) and (d), we conclude that

$$a \approx 0{,}7A \cong \frac{1+\nu}{E}\sigma_s, \qquad (3.15)$$

where E is Young's modulus, ν is Poisson's constant, and σ_s is the yield point*.

The difference between the value of a found here and that found in the preceding Section, as was to be expected, is insignificant.

If we retain the components of the small-deformation tensor in the conditions (2.16), then the quantity a will not be connected with the quantities Ω_{ik}. These values, however, are still restricted by certain conditions, to be mentioned below.

Again, eqs.(2.6) permit us to find λ^* and μ^*. We obtain

$$\lambda^* = \lambda(1 + A_{11}) + \mu A_{12}, \qquad (3.16a)$$

$$\mu^* = \lambda A_{21} + \mu(1 + A_{22}). \qquad (3.16b)$$

where the quantities A_{ik} are functions of the quantities a, α, β and of the antisymmetric tensor components Ω_{ik}. The form of these functions is determined by the composition of the coefficients b_{ik} and b_i.

Since in λ^* and μ^* there enter the parameters Ω_{ik}, which depend on unknown components of the displacement vector, the formulas (3.16a) - (3.16b) cannot be directly used for substitution into eqs.(2.14) - (2.15c). To construct the first approximation to the solution of the nonlinear problem of the theory of elasticity, the obtained expressions for λ^* and μ^* must be averaged over Ω_{ik}, and this will be discussed below.

A comparison of the coefficients b_i, determined by eqs.(2.10a) - (2.10b) and (3.12a) - (3.12b), shows that the presence of finite rotations of the elements of the body has a noticeable effect on the properties of the approximations being considered. Of importance is likewise the choice of the interval (α, β). At certain values of the quantities Ω_{ik}, a, α, β, the right-hand sides of eqs.(2.6) may vanish, while at other values of these quantities, the

* See the above-cited book by Il'yushkin, pp.98-100; we have $A = \frac{3}{\sqrt{2}} e_i$; $e_i = \frac{\sigma_i}{3G}$, whence follows eq.(3.15).

functions A_{ik} may take negative values. For $b_i = 0$, obviously, the conventional linearization is permissible, consisting in a substitution of the tensor components D_{ik} by the tensor components ϵ_{ik}.

For negative values of the functions A_{ik}, there is a local diminution of the reduced elastic constants λ^* and μ^* by comparison with λ and μ, which is interpreted as a local decrease in the rigidity of the material. For positive A_{ik}, the rigidity of the material undergoes an apparent increase. Consequently, the nonlinearity of the components of the tensor D_{ik} leads to /190 the development of a quasi-inhomogeneity of the mechanical properties of the material, which may be called kinematic.

Of course, the above statements can merely be regarded as certain heuristic conclusions which require more detailed justification. For this reason, we shall present additional explanations.

1. Preliminary Selection of the Region of Approximate Representations of the Potential Energy Π by the Energy Π^*

The choice of the region ω is of fundamental significance. This is known from the theory of approximation functions but is also clear from the preceding argument. If we dispose arbitrarily of the quantities α, β and Ω_{ik}, we may evidently impart almost any desired values to the functions A_{ik} and reduce the solution of the problem to a physical absurdity.

Concrete problems of mechanics disclose relations between the quantities ϵ_{ik} and Ω_{ik}. For this reason, by prescribing the interval (α, β) over which the quantities ϵ_{ik} vary, we also impose certain restrictions on the region of variation of the quantities Ω_{ik}. The difficulty is that these restrictions are not prescribed in advance in the form of explicit analytic relations. In other words, there exists a correlation of the quantities ϵ_{ik} and Ω_{ik}, but the limits of variation of the correlation factor, differing from the trivial cases of zero and unity, are unknown*. We can only assert that, as the region of variation of ϵ_{ik} and Ω_{ik} is expanded, the probability increases for points corresponding to the physical relation between ϵ_{ik} and Ω_{ik} to fall within this region.

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* The reader may raise the question whether it is legitimate to introduce the concept of correlation here. Indeed, in solving concrete boundary problems of the elasticity theory, a direct connectivity is established between the values of ϵ_{ik} and Ω_{ik}, i.e., in this case the coefficient of correlation is unity. But the very essence of the problem of approximation under consideration here is precisely that this approximation does not rely on the solution of any partial problem. Imagine the set of possible deformed states of a body described by the tensors ϵ_{ik} and Ω_{ik}. To each state there corresponds a point in the six-dimensional region of the quantity ϵ_{ik} and the three-dimensional region of the components of Ω_{ik}. If we do not know the analytic connection between the points in these spaces, then the correlational connectivity comes into force. The coefficient of correlation characterizes the probability of the physical correspondence of point A of the first space to point B of the second space.

On the basis of the above statements, we shall not consider approximation over an arbitrarily small interval. We shall instead consider a finite region in the six-dimensional space of the quantities Ω_{ik}, bounded by the condition:

$$|\beta-\alpha|\geqslant 1. \tag{3.17}$$

For example, as stated above, let us put $\beta = 1$, $\alpha = 0$ or $\beta = 0$, $\alpha = -1$. /191

The choice of a sufficiently wide interval of variation of the components of ϵ_{ik} eliminates the need for establishing an exact connectivity between ϵ_{ik} and Ω_{ik}.

2. Preliminary Delimitation of the Region of Variation of the Quantities Ω_{ik}

As we know from the theory of deformation of thin rods, plates, and shells, finite displacements and rotations may simultaneously arise in their elements under small deformations ε_{ik}. As already mentioned, there is a correlation between the quantities ϵ_{ik} and Ω_{ik}. Since the correlation coefficient is unknown, a region of variation of the quantities Ω_{ik} is assigned arbitrarily at first; for example, it is assumed that these quantities vary from zero to $\pm b$ where the quantity b is at first arbitrarily prescribed. Then we investigate λ^* and μ^* at various values of ϵ_{ik} lying on the interval (αa, βa) and various values of Ω_{ik}, lying on the interval (-b, +b). The value of the parameter b is restricted by the requirement that λ^* and μ^* shall be positive and by the requirement that the Poisson constants ν^* shall be included in the interval (0; 0.5).

On preliminary determination of the region of variation of the quantities Ω_{ik}, we may use experimental data, for example the results of a study on the deformation of shells under great displacements and angles of rotation in the supercritical stage.

The study of the variation of λ^* and μ^* in the four-dimensional region (αa, βa; Ω_{ik}) is in itself a means of the qualitative analysis of special problems.

3. Determination of the Mean Values of λ^* and μ^*

Establishing first the variational region Ω_0 of the components Ω_{ik}, let us average the quantities λ^* and μ^* in this region. We find

$$\bar{\lambda}^*=\frac{1}{\Omega_0}\int\limits_{(\Omega_0)}\lambda^* d\Omega_{12}d\Omega_{23}d\Omega_{31}; \tag{3.18a}$$

$$\bar{\mu}^*=\frac{1}{\Omega_0}\int\limits_{(\Omega_0)}\mu^* d\Omega_{12}d\Omega_{23}d\Omega_{31}. \tag{3.18b}$$

This averaging may be done with the weight $p(\Omega_{ik})$ if it is possible to indicate a function $p(\Omega_{ik})$ on the basis of experiments or theoretical considerations. Then, eqs.(3.18a) - (3.18b) are replaced by the following: /192

$$\bar{\lambda}^* = \frac{\int\limits_{(\Omega_0)} p(\Omega_{ik}) \lambda^* d\Omega_{12} d\Omega_{23} d\Omega_{31}}{\int\limits_{(\Omega_0)} p(\Omega_{ik}) d\Omega_{12} d\Omega_{23} d\Omega_{31}}; \tag{3.19a}$$

$$\bar{\mu}^* = \frac{\int\limits_{(\Omega_0)} p(\Omega_{ik}) \mu^* d\Omega_{12} d\Omega_{23} d\Omega_{31}}{\int\limits_{(\Omega_0)} p(\Omega_{ik}) d\Omega_{12} d\Omega_{23} d\Omega_{31}}. \tag{3.19b}$$

By a change in scale, the integrals $\int\limits_{(\Omega_0)} \ldots d\Omega_{ik}$ can always be reduced to integrals within limits lying inside the interval (-1, 1).

The quantities λ^* and μ^* are introduced into eqs.(2.14) - (2.15c), and thereby we complete the solution of the problem of linear approximation, in first approximation.

Section 3a. Further Development of the Method of Linear Approximation

As noted in Sect. 3, the determination of the region of variation of the tensor components, over which the approximate representation extends, is of fundamental importance in performing the approximation of components of the finite-deformation tensor by components of the small-deformation tensor.

In Sect.3, we assumed that the basic region was a six-dimensional space with the coordinates ϵ_{ik}, and as an auxiliary region the three-dimensional space with the coordinates Ω_{ik}. We shall now supplement the above.

If we know in advance, from the conditions of the problem, that certain components of the tensor ϵ_{ik} or Ω_{ik} are zero, then the number of dimensions of the basic and auxiliary regions is correspondingly decreased. This clearly leads to obvious changes in the multiplicity of the integrals entering into the formulas of Sect.3.

We assumed that all the quantities ϵ_{ik} vary over the total interval $(\alpha a, \beta b)$. On the basis of the contents of Sect.2, we may consider a more general case, individualizing the variational interval for each quantity of ϵ_{ik}.

Let us place the components ϵ_{11}, ϵ_{22}, ϵ_{33}, ϵ_{12}, ϵ_{23}, and ϵ_{31} in corre-

spondence with the numbers 1, 2, 3, 4, 5, 6. Let the components of ϵ_{ik} vary over the intervals $(\alpha_j a_{ik}, \beta_j a_{ik})$ where the symbols j correspond to pairs of numbers i, k in the above-discussed manner. Instead of eqs.(3.3) we introduce the relations /193

$$\varepsilon_{ik} = a_{ik}\eta_{ik}. \tag{3a.1}$$

The transformation of the basic relations described in Sect.3 is obvious in this case, and we will not repeat them here. We note that the introduction of individual variational integrals of the components of ϵ_{ik} is possible only in those special cases where the condition of the problem of mechanics permits such individualization. In exactly the same way, separate variational intervals of the components of the antisymmetric tensor Ω_{ik} may be introduced.

Let us continue our consideration of the further development of the proposed method. Assume that we have solved the quasi-linear dynamic problem of the theory of elasticity with the constants $\overline{\lambda}^*$ and $\overline{\mu}^*$. This quasi-linear solution gives a first approximation to the solution of the nonlinear problem in displacements. Then, the tensor components Ω_{ik}, in first approximation, will be known functions of the coordinates $x^j (j = 1, 2, 3)$ of an elastic body and of the time t. Returning to eqs.(3.16a) - (3.16b), we find λ^* and μ^* as functions of x^j and t. Substituting the resultant values of λ^* and μ^* into eqs.(2.14) and determining the tensor components ϵ_{ik} from the first approximation, we find the first approximation for the field of stresses. The first approximation for the stresses will contain nonlinear terms depending on Ω_{ik}.

On the basis of the first approximation, the total variational interval of the quantities ϵ_{ik} can also be refined. However, a considerable decrease in the diameter of the region ω^* may lead to the contradictions mentioned above.

We recall now that the linearization of the components of D_{ik} still does not result in a complete linearization of the equations of elastodynamics, since other sources of nonlinear equations considered in Chapter II are of considerable significance here. In this connection, we must again investigate the approximate method of linearization in a somewhat more complex form, and then go on to obtaining further approximation.

Section 4. Linearization in an Element of the Shell

The results of the preceding Sections permit an approximate elimination of the nonlinear terms entering into the equations expressing Hooke's law on introduction of the components of the finite-deformation tensor into these equations. This, however, does not lead to linear equations of motion of an element of the shell, since the components of the strain tensor enter into the expression $\sqrt{G}dx^1dx^2dx^3$ for the volume of the deformed element.

We are now confronted by the following alternative: either to retain the approximate linear expressions (2.14) obtained above for the stress tensor /194 components connected with the specific energy of deformation and not to carry the linearization to completion, or else to consider the quasi-specific energy

of deformation determined by the equation

$$V = \Pi \sqrt{G}, \tag{4.1}$$

and, by introducing variations into the results above, obtain a complete linearization, and in this case to enter into formal contradiction with the well-known energetic principles of Hooke's law.

It seems permissible to make use of eqs.(4.1), since every approximate solution of a physical problem contains errors contradicting the exact solutions.

Consider the integral

$$I = \int_{(\Omega)} [\Pi^* \sqrt{g} - \Pi \sqrt{G}]^2 \, d\omega. \tag{4.2}$$

Making use of (II, 6.3) and retaining in the integral expression all terms of the order of $(\varepsilon_{ik})^4$, we obtain

$$I \cong \sqrt{g} \int_{(\Omega)} \left\{ \Pi^* - \Pi \left[1 + g^{ik} D_{ik} + \frac{1}{g} \delta^{irp} \delta^{ksq} g_{pq} \varepsilon_{ik} \varepsilon_{rs} - \right.\right.$$

$$\left.\left. - \frac{1}{2} (g^{ik} \varepsilon_{ik})^2 \right] \right\}^2 d\Omega. \tag{4.3}$$

From eqs.(2.3b), it follows that

$$g^{ik} D_{ik} = g^{ik} \varepsilon_{ik} + \frac{1}{2} g^{ik} g^{jp} x_{ip} x_{kj}. \tag{a}$$

Let us represent I in expanded form:

$$I \cong \sqrt{g} \int_{(\Omega)} \left\{ \frac{1}{2} (\lambda^* - \lambda) (g^{rs} \varepsilon_{rs})^2 + (\mu^* - \mu) g^{ir} g^{ks} \varepsilon_{ik} \varepsilon_{rs} - \right.$$

$$- \frac{1}{2} \lambda \left[(g^{rs} \varepsilon_{rs}) (g^{rs} g^{jp} x_{rp} x_{sj}) + \frac{1}{4} (g^{rs} g^{jp} x_{rp} x_{sj})^2 \right] -$$

$$- \mu \left[g^{ir} g^{ks} g^{jp} \varepsilon_{ik} x_{rp} x_{sj} + \frac{1}{4} g^{ir} g^{ks} g^{qm} g^{jp} x_{im} x_{kq} x_{rp} x_{sj} \right] -$$

$$-\frac{1}{2}\lambda\left[(g^{rs}\varepsilon_{rs})^3+\frac{3}{2}(g^{rs}\varepsilon_{rs})^2(g^{rs}g^{jp}x_{rp}x_{sj})-\frac{1}{2}(g^{rs}\varepsilon_{rs})^4+\right.$$

$$\left.+\frac{1}{g}(g^{rs}\varepsilon_{rs})^2\delta^{irp}\delta^{ksq}g_{pq}\varepsilon_{ik}\varepsilon_{rs}\right]-\mu\left[(g^{rs}\varepsilon_{rs})g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs}+\right.$$

$$+(g^{rs}\varepsilon_{rs})g^{ir}g^{ks}g^{jp}\varepsilon_{ik}x_{rp}x_{sj}+\frac{1}{2}(g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs})(g^{ik}g^{jp}x_{ip}x_{kj})+$$

$$\left.\left.+\frac{1}{g}(g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs})(\delta^{irp}\delta^{ksq}g_{pq}\varepsilon_{ik}\varepsilon_{rs})-\frac{1}{2}(g^{rs}\varepsilon_{rs})^2g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs}\right]\right\}^2 d\Omega. \tag{4.4}$$

On comparing eqs.(4.4) and (2.5) we conclude that only the right-hand sides of eqs.(2.6) are changed. The new right-hand sides of eqs.(2.6) now have the following form: /195

$$B_1=b_1+c_1;\; B_2=b_2+c_2. \tag{4.5}$$

It is clear from eqs.(4.4) that the c_i are expressed as follows:

$$c_1=\frac{1}{2}\int\limits_{(\Omega)}(g^{rs}\varepsilon_{rs})^2(\lambda\psi_1+2\mu\psi_2)\,d\Omega;\qquad c_2=\int\limits_{(\Omega)}g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs}(\lambda\psi_1+2\mu\psi_2)\,d\Omega. \tag{4.6}$$

where

$$\psi_1=(g^{rs}\varepsilon_{rs})_2\left[g^{rs}\varepsilon_{rs}+\frac{3}{2}g^{rs}g^{jp}x_{rp}x_{sj}-\frac{1}{2}(g^{rs}\varepsilon_{rs})^2+\frac{1}{g}\delta^{irp}\delta^{ksq}g_{pq}\varepsilon_{ik}\varepsilon_{rs}\right]; \tag{4.7a}$$

$$\psi_2=(g^{rs}\varepsilon_{rs})g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs}+(g^{rs}\varepsilon_{rs})g^{ir}g^{ks}g^{jp}\varepsilon_{ik}x_{rp}x_{sj}+\frac{1}{2}(g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs})\times$$

$$\times(g^{ik}g^{jp}x_{ip}x_{kj})+\frac{1}{g}(g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs})(\delta^{irp}\delta^{ksq}g_{pq}\varepsilon_{ik}\varepsilon_{rs})-$$

$$-\frac{1}{2}(g^{rs}\varepsilon_{rs})^2g^{ir}g^{ks}\varepsilon_{ik}\varepsilon_{rs}. \tag{4.7b}$$

As above, we now pass to a local system of rectangular Cartesian coordinates, taking

$$g_{ii}=1;\; g_{ik}=0\qquad (i\neq k). \tag{b}$$

To transform the expression c_1, we must make use of the relations employed in transforming the expressions for ψ_1 and ψ_2 in the last Section, and of the formula

$$\delta^{irp}\delta^{ksq}g_{pq}\varepsilon_{ik}\varepsilon_{rs}=a^2\left(\sum_{i=1}^{3}\eta_{ii}\right)^2-a^2\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2. \tag{4.8}$$

We find

$$c_1=\frac{1}{2}a^8\int\limits_{(\omega)}\left(\sum_{i=1}^{3}\eta_{ii}\right)^2(\lambda\psi_1+2\mu\psi_2)\,d\omega, \tag{4.9a}$$

$$c_2=a^8\int\limits_{(\omega)}\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\eta_{ik}^2\right)(\lambda\psi_1+2\mu\psi_2)\,d\omega. \tag{4.9b}$$

where

$$d\omega=d\eta_{11}\ldots d\eta_{23}.$$

After transformations, Ψ_1 and Ψ_2 take the following form: /196

$$\psi_1=a^2\left(\sum_{i=1}^{3}\eta_{ii}\right)^2\left[\frac{3}{2}A^2+a\sum_{i=1}^{3}\eta_{ii}+\frac{1}{2}a^2\left(\sum_{i=1}^{3}\eta_{ii}\right)^2+\frac{1}{2}a^2\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right]; \tag{4.10a}$$

$$\psi_2=a^2\left[\frac{1}{2}A^2\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right)+\left(\sum_{i=1}^{3}\eta_{ii}\right)\left(\sum_{j=1}^{3}\sum_{k=1}^{3}B_{jk}\eta_{jk}\right)\right]+a^3\left(\sum_{i=1}^{3}\eta_{ii}\right)\times$$
$$\times\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right)+a^4\left[\left(\sum_{i=1}^{3}\eta_{ii}\right)\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\xi_{jk}\eta_{jk}\right)-\frac{1}{2}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right)^2+\right.$$
$$\left.+\frac{1}{2}\left(\sum_{i=1}^{3}\eta_{ii}\right)^2\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right)\right]. \qquad (4.10^b) \tag{4.10b}$$

By analogy to eqs.(3.12a) - (3.12b), we obtain

$$c_1=\frac{1}{2}a^8\left[\lambda(e_{12}a^2A^2+e_{13}a^3+e_{14}a^4)+2\mu\left(f_{12}a^2A^2+\right.\right.$$

$$+a^2\sum_{j=1}^{3}\sum_{k=1}^{3}B_{jk}\vartheta_{jk}+f_{13}a^3+f_{14}a^4\Big)\Big]; \tag{4.11a}$$

$$c_2=a^8\Big[\lambda(e_{22}a^2A^2+e_{23}a^3+e_{24}a^4)+2\mu\Big(f_{22}a^2A^2+$$

$$+a^2\sum_{j=2}^{3}\sum_{k=1}^{3}B_{jk}\theta_{jk}+f_{23}a^3+f_{24}a^4\Big)\Big] \tag{4.11b}$$

The coefficients entering into these formulas are expressed by the following sextuple integrals:

$$e_{12}=\frac{3}{2}\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^4 d\omega;\quad e_{13}=\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^5 d\omega;$$

$$e_{14}=\frac{1}{2}\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^2\Big[\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^4+\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^2\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)\Big]d\omega;$$

$$e_{22}=\frac{3}{2}\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^2\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)d\omega;$$

$$e_{23}=\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^3\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)d\omega; \tag{4.12a}$$

$$e_{24}=\frac{1}{2}\int\limits_{(\omega)}\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)\Big[\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^4+\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^2\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)\Big]d\omega;$$

$$f_{12}=\frac{1}{2}\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^2\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)d\omega;\ f_{13}=\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^3\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)d\omega;$$

$$f_{14}=\int\limits_{(\omega)}\Big(\sum_{i=1}^{3}\eta_{ii}\Big)^2\Big[\Big(\sum_{i=1}^{3}\eta_{ii}\Big)\sum_{j=1}^{3}\sum_{k=1}^{3}\xi_{jk}\eta_{jk}-\frac{1}{2}\Big(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\Big)^2+$$

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$$+\frac{1}{2}\left(\sum_{i=1}^{3}\eta_{ii}\right)^2\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}\right)^2\Bigg]\,d\omega;$$

$$f_{22}=\frac{1}{2}\int\limits_{(\omega)}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right)^2 d\omega;\quad f_{23}=\int\limits_{(\omega)}\left(\sum_{i=1}^{3}\eta_{ii}\right)\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right)^2 d\omega;$$

$$f_{24}=\int\limits_{(\omega)}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right)\Bigg[\left(\sum_{i=1}^{3}\eta_{ii}\right)\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\xi_{jk}\eta_{jk}\right)-\frac{1}{2}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right)^2+$$

$$+\frac{1}{2}\left(\sum_{i=1}^{3}\eta_{ii}\right)^2\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right)\Bigg]\,d\omega;$$

(4.12b)

$$\vartheta_{jk}=\int\limits_{(\omega)}\left(\sum_{i=1}^{3}\eta_{ii}\right)^3\eta_{jk}\,d\omega; \qquad (4.12c)$$

$$\theta_{jk}=\int\limits_{(\omega)}\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\eta_{jk}^2\right)\left(\sum_{i=1}^{3}\eta_{ii}\right)\eta_{jk}\,d\omega. \qquad (4.12d)$$

As in Sect.3, we may substitute in relations (4.9a) - (4.9b) and in the resultant expressions (4.12a) - (4.12d):

$$\int\limits_{(\omega)}\ldots d\omega=\int\limits_{\alpha}^{\beta}\ldots\int\limits^{\beta}\ldots d\eta_{11}\ldots d\eta_{23}.$$

In some cases, as noted in Sect.3a, the intervals of variation of the variables η_{ik} must be individualized. Then, /198

$$\int\limits_{(\omega)}\ldots d\omega=\int\limits_{\alpha_1}^{\beta_1}\ldots\int\limits_{\alpha_6}^{\beta_6}\ldots d\eta_{11}\ldots d\eta_{23}.$$

An example of approximation over various variational intervals of η_{ik} is given in Sect.6.

Instead of eqs.(3.16a) - (3.16b), we obtain

$$\lambda^* = \lambda(1 + c_{11}) + \mu c_{12}, \tag{4.13a}$$

$$\mu^* = \lambda c_{21} + \mu(1 + c_{22}). \tag{4.13b}$$

The properties of the coefficients c_{ik} are analogous to those of the coefficients A_{ik} in eqs.(3.16a) - (3.16b) considered in Sect.3.

We give below an example illustrating the contents of Sect.4.

Section 5. On the Relation between Linear Approximation of the Components of the Finite-Deformation Tensor and the Method of Equivalent Linearization and the Probability Methods. Further Stages of Successive Approximations

The above method of linearization for the stress tensor components and the finite-deformation tensor components are close to the methods of equivalent linearization, which we know from the nonlinear mechanics of systems with a finite number of degrees of freedom*.

The method applied above is also closely related to the probability method of solving the problems of mechanics. In fact, the region Ω_4 consisting of the regions ω and Ω_0 in which the approximation is performed, is the region of "probable states" of an elastic body. The method of approximation adopted by us is equivalent to the hypothesis that these states are equiprobable, which only approximately corresponds to reality.

In investigating narrow classes of problems of the mechanics of shells, one must bear in mind the results of experiments permitting us to construct functions characterizing the probability distribution of the appearance of certain values of the quantities forming the regions ω and Ω_0. By using prob- /199
ability distribution curves, we must introduce functions of weight into the integrals I of the preceding Sections, i.e., we must consider the weighted-square deviations. The quantities a_{ik} entering into eqs.(3a.1) are special forms of the weighting functions.

We do not dispose of the necessary experimental data and were therefore compelled to abstain from the use of weighted-square deviations.

In the development of Section 3 - 4 we give a summary of the first stages

* Various versions of the method of equivalent linearization can be found by the reader in the following books: N.N.Bogolyubov, Yu. A. Mitropol'skiy, Asymptotic Methods in the Theory of Nonlinear Vibrations, Fizmatgiz, 1958; S.Krendell, Random Vibrations of Systems with Nonlinear Restoring Forces, Transactions of the Symposium on Nonlinear Vibrations, Kiev, 1961; Ya.G.Panovko, Action of Periodic Impacts on a Nonlinear Elastic System with One Degree of Freedom, Trudy In-ta Fiziki AN Latv. SSR, No.5, 1953

of the construction of approximate solutions for the nonlinear boundary problems of the theory of shells, based on the use of the linear approximation method of the components of the finite-deformation tensor.

Let us consider only those problems in which the nonlinearity is connected with the existence of finite displacements and angles of rotation. The components of the strain tensor ϵ_{ik} will be regarded as quantities small in comparison with unity.

Turning to the general equation of dynamics (III,15.1), we note that the linear approximation, with an introduction of averaged elastic constants $\bar{\lambda}^*$ and $\bar{\mu}^*$ considered in the preceding Sections, makes it possible to linearize the operators entering into the elementary work of the internal forces δA. The terms expressing the work of the body and surface forces and of the inertial forces will, as before, still contain nonlinear summands. But these nonlinear terms will not contain components of Ω_{ik} but will depend instead on the scalar $g^{ik}\epsilon_{ik}$ if we disregard all the second-order terms in the composition of D_{ik}, which after multiplication by the inertial forces will yield terms with a homogeneity index equal to three with respect to the displacement components and their derivatives. If, in agreement with most investigators, we neglect the influence of the volumetric expansion $g^{ik}\epsilon_{ik}$ on the virtual work of the inertial forces, the body forces and the surface forces, then the approximation given in Sects.3-4 will permit a linearization of the system of equations of motion*. The resultant system, in accordance with Sects.3-3a, will be denoted as a system of equations of first approximation. We note that this system of equations will differ from the systems of linear equations of Chapter III, obtained by direct rejection of all nonlinear terms. The difference will depend on the term $\alpha g_{ik} g^{rs}\epsilon_{rs}$ entering into the composition of D_{ik} according to eq.(2.15b).

We will not further discuss the construction of this system, since its method of derivation does not differ in principle from that discussed in Chapter III.

After deciding to linearize the boundary problem in displacements, we will obtain the field of stresses, making use of eqs.(2.14) and (4.13a) - (4.13b), /200
together with eqs.(2.6) and their coefficients, found in Sect.3-4.

As already noted in Sect.3a, the resultant expressions for the stress tensor components will contain nonlinear terms depending on the components of the antisymmetric tensor Ω_{ik}. Supplementing our remarks in Sect.3a, it will be recalled that the expressions for λ^* and μ^* in Sects.2 - 4 were obtained in a local Cartesian coordinate system. For this reason, in considering the field of stresses, we must transform the components of Ω_{ik} into the local Cartesian system of coordinates, and only then determine the quantities of λ^* and μ^*

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* The terms of the order of ϵ_{ik} and of higher orders in the expression for the virtual work of the forces of inertia and the living forces are neglected in the equations of the monograph (Bibl.12). Returning to the questions considered here, we note that the approximation given in Sect.3 can be applied in this case.

As a result, we obtain the first approximation for the stress tensor components. To obtain the second approximation, let us substitute into eqs.(2.14) the values of λ^* and μ^* found from the first approximation. We recall that these values for λ^* and μ^* in the problems of shell mechanics are functions of the coordinates x^i of the points of its basic surface and also of the z coordinates. When substituting, in the variational equation, all nonlinear terms which had been neglected in obtaining the first approximation, by quantities found from the solutions of the equations of first approximation, we obtain the system of linear equations of second approximation. Such a system of linear equations will have variable coefficients. This complicates the solution of the problems, but, evidently permits a qualitative analysis of the solutions of the nonlinear problem, that is more profound than an analysis based on the equations of second approximation obtained by the ordinary methods of the elasticity theory. An example of these methods is the use of the nonlinear Lamé equations (II, 7.5a) - (II, 7.5b), where additional body forces ξ^i are determined from the first approximation. We have already noted the disadvantages of such methods (II, Sect.8).

To facilitate the introduction of the methods considered in Sects.3-4, we present an illustrative example in the following Section*. This example will also permit us to supplement the general characterization of the significance of the method.

Section 6. On Axisymmetric Deformations and the Elastic Stability of a Circular Tube Subjected to the Action of Longitudinal Compressive Forces

The heading of this Section coincides with the title of another work (Bibl.23d). Here we make use of certain results of that paper, with the object of their further development on the basis of the method given in Sects.3-4. At the same time, the method of approximation under consideration is given /201
an elementary illustration, showing its promise.

We consider the well-known problem of the stability of a circular cylindrical tube, compressed by longitudinal forces uniformly distributed along the contour of the basic (middle) surface in the face sections of the cylinder.

We shall study only the case of axisymmetric deformations, although the experimental and theoretical results obtained in the last 15 years convincingly prove the significance not of purely axisymmetric deformations, but of deformations with a cyclic symmetry about the axis (Bibl.4, 10). The assumption of the possible existence, in this case, of axisymmetric forms of deformation in the transcritical stage is likewise confirmed by experiments described in older reports. Evidently, in this work relatively thick shells were investigated, and the symmetric forms of deformation were accompanied by stresses exceeding the yield point of the material**. Thus, we shall confine ourselves to a con-

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* This example cannot be the object of the studies in Sects.3-4, since the extreme simplicity of the problem admits many more solutions than that given below.

**Cf. I.V.Gekkeler, Statics of an Elastic Body, ONTI, 1934, pp.271-276; S.P.Timoshenko, Stability of Elastic Systems, OGIZ, 1946, pp.388-392

sideration of the axisymmetric deformations of a closed circular cylindrical shell. Following the notation adopted earlier (Bibl.23d), we put

$$x^1 = x;\ x^2 = s;\ x^3 = z, \tag{6.1}$$

where the coordinate x is the distance from one of the face contours of the undeformed middle surface measured along the generatrix, and s is the length of arc of the directrix measured from some initial point, while the z coordinate is measured along a normal to the undeformed middle surface in the direction toward the axis of the tube. In longal coordinates (6.1) on the undeformed middle surface, we have

$$g_{11} = g_{22} = g_{33} = 1;\quad g_{ik} = 0 \quad (i \neq k). \tag{6.2}$$

Making use of the method of successive approximations given in (III, Sects.9-10), we confine ourselves here to the first approximation. Accordingly, we put

$$u_1 = u + zu^{(1)};\quad u_2 = v + zv^{(1)};\quad u_3 = w + zw^{(1)}, \tag{6.3}$$

where u, v, w are the displacement vector components of a point of the middle surface of the shell.

As we know from Chapter III, the expression (6.3) corresponds in accuracy to the reduction formulas resulting from the Kirchhoff-Love hypotheses. An increase in the accuracy of reduction is not necessary since this example only has an illustrative purpose.

In axisymmetric deformations, the displacement vector components of a point of the middle surface are expressed as follows: /202

$$u = u(x);\ v = 0;\ w = w(x). \tag{a}$$

Using (III, 9.1), (III, 9.2), we find

$$u^{(1)} = -\frac{dw}{dx};\quad v^{(1)} = 0;\quad w^{(1)} = -\frac{\lambda}{\lambda + 2\mu}\left(\frac{du}{dx} - kw\right), \tag{6.4}$$

where $k = R^{-1}$ is the curvature of a section of the middle surface normal to its generatrix. At the accuracy for constructing the equations adopted by us, these same relations are equivalent to the conditions

$$\varepsilon_{31} = \varepsilon_{23} = 0; \quad \varepsilon_{33} = w^{(1)}. \tag{6.5}$$

Using (III, 10.1a) - (III, 10.3c), we find

$$\varepsilon_{11} = \frac{du}{dx} - z\frac{d^2w}{dx^2}; \quad \varepsilon_{22} = -kw; \quad \varepsilon_{12} = 0. \tag{6.6}$$

If we remain within the limits of the accuracy adopted by us, then we may approximately put

$$\varepsilon_{33} \cong -\frac{\lambda}{\lambda + 2\mu}(\varepsilon_{11} + \varepsilon_{22}). \tag{6.7}$$

Thus, of the six strain tensor components, three vanish in this case, and the component ε_{33}, neglecting the summands of order z in its composition, is expressed in terms of ε_{11} and ε_{22}.

Consequently, the linear approximation of the components of the finite-deformation tensor, in the problem under consideration, must be performed in the two-dimensional space of the quantities ε_{11} and ε_{22}. Of course, the restriction in the number of space dimensions of possible deformations was obtained here as a result of very gross simplifications. We have given above the motivation for this approach to the problem.

On the basis of eqs.(6.3), we find

$$\Omega_{12} = \Omega_{23} = 0; \quad \Omega_{31} = \frac{1}{2}\left(\frac{dw}{dx} - u^{(1)} + z\frac{dw^{(1)}}{dx}\right), \tag{6.8a}$$

or, bearing in mind eqs.(6.4),

$$\Omega_{31} = \frac{dw}{dx} - \frac{1}{2}\frac{\lambda}{\lambda + 2\mu}z\left(\frac{d^2u}{dx^2} - k\frac{dw}{dx}\right). \tag{6.8b}$$

Hereafter we shall consider only the mean value of Ω_{31} over the thickness of the shell, expressed by the formula

$$\bar{\Omega}_{31} = \frac{dw}{dx}. \tag{6.9}$$

From eq.(3.7b), we find /203

$$A^2 = 2(\bar{\Omega}_{31})^2. \tag{6.10}$$

Equations (3.8a) - (3.8b) yield

$$B_{11} = \frac{1}{4}(\bar{\Omega}_{31})^2; \quad B_{33} = \frac{1}{4}(\bar{\Omega}_{31})^2. \tag{6.11a}$$

The other functions B_{ik} vanish. Further,

$$C^2 = \frac{1}{32}(\bar{\Omega}_{31})^4. \tag{6.11b}$$

Using eqs.(3.12a) - (3.12b), (4.11a) - (4.11b), and (6.10) - (6.11b), we find

$$\begin{aligned}\lambda^* = \lambda + \lambda a^{-2}[m_{10}(\bar{\Omega}_{31})^4 + m_{11}a(\bar{\Omega}_{31})^2 + m_{12}a^2(\bar{\Omega}_{31})^2 + m_{13}a^3 + \\ + m_{14}a^4] + \mu a^{-2}[n_{10}(\bar{\Omega}_{31})^4 + n_{11}a(\bar{\Omega}_{31})^2 + n_{12}a^2(\bar{\Omega}_{31})^2 + n_{13}a^3 + n_{14}a^4];\end{aligned} \tag{6.12a}$$

$$\begin{aligned}\mu^* = \mu + \lambda a^{-2}[m_{20}(\bar{\Omega}_{31})^4 + m_{21}a(\bar{\Omega}_{31})^2 + m_{22}a^2(\bar{\Omega}_{31})^2 + m_{23}a^3 + \\ + m_{24}a^4] + \mu a^{-2}[n_{20}(\bar{\Omega}_{31})^4 + n_{21}a(\bar{\Omega}_{31})^2 + n_{22}a^2(\bar{\Omega}_{31})^2 + n_{23}a^3 + n_{24}a^4].\end{aligned} \tag{6.12b}$$

The quantities λ^* and μ^* determine the required linear approximation in the special case under consideration.

In calculating the coefficients m_{ik} and n_{ik}, eqs.(3.13a) - (3.13d) and (4.12a) - (4.12d) must be used, bearing in mind eqs.(6.5) - (6.7) and passing into the two-dimensional region of integration according to the indications given in Sect.3a.

The radial displacements w of the middle surface establish a field of flexural stresses. Therefore, the shell is divided into two zones over its thickness, a zone in which the tensile stresses dominate and a zone in which the compressive stresses σ_{11} dominate. In the zone of tensile stresses, the variable η_{11} in the integrals (3.13a) - (3.13d) and (4.12a) - (4.12d) varies over the interval (0,1). In the zone of compressive stresses, this variable varies over the interval (-1,0). We assume that the variable η_{22}, corresponding to the component ϵ_{22} of the strain tensor, varies over the symmetric interval (-1,1).

The numerical values of the coefficients m_{ik} and n_{ik} here depend on λ and μ. This relation is due to the fact that ϵ_{33} is expressed in terms of ϵ_{11} and ϵ_{22} by eq.(6.7). Consequently,

$$\eta_{33} = -\frac{\lambda}{\lambda + 2\mu}(\eta_{11} + \eta_{22}).$$

The ratio $\lambda : (\lambda + 2\mu)$ is independent of Young's modulus E, but depends only on Poisson's constant ν. Table 1 shows the physical constants and parameters a for steel, aluminum, and duralumin. We have assumed that the constants ν for these materials are the same, and therefore the coefficients m_{ik} and n_{ik} for these materials, shown in Table 2, are also the same. /204

TABLE 1

No.	Material	$E \cdot 10^{-6}$, bar	ν	σ_s, bar	$a \cong \frac{1+\nu}{E}\sigma_s$	$a^{-2} \cdot 10^{-4}$
1	STEEL 30KhGSA . .	2,1	0,3	12700	$0{,}805 \cdot 10^{-2}$	1,54
2	ALUMINUM AMG . .	0.68	0,3	2060	$0{,}396 \cdot 10^{-2}$	6,38
3	DURALUMIN D17 .	0,70	0,3	2350	$0{,}439 \cdot 10^{-2}$	5,19

TABLE 2

i	m_{ik}					n_{ik}				
	0	1(±)	2	3(±)	4	0	1(±)	2	3(±)	4
1	−0,626	2,67	9,07	2,99	1,51	−0,0393	−0,703	13,0	6,06	−0,812
	0,542	−0,0430	−0,792	−0,0895	0,101	0,0341	0,180	−0,827	−0,129	0,0929

In the columns with a (±) sign, the numerical values of the coefficients are given for the region of tension. In the region of compression these coefficients have the opposite sign.

It will be seen from eqs.(6.12a) - (6.12b) that the quantities λ^* and μ^* in the zone of tensile stresses are always greater, respectively, than λ and μ. In the zone of compressive stresses this increase of λ^* and μ^* over λ and μ may also occur, but the differences $\lambda^* - \lambda$ and $\mu^* - \mu$ are smaller than in the first zone, since the quantities in odd powers of the parameter a are negative.

If the parameter a and the components of Ω_{31} are such that the trinomials

$$m_{i0}(\bar{\Omega}_{31})^2 + m_{i1}a + m_{i2}a^2 \text{ and } n_{i0}(\bar{\Omega}_{31})^2 + n_{i1}a + n_{i2}a^2$$

in the zone of compression are negative, then in this zone the differences $\lambda^* - \lambda$ and $\mu^* - \mu$ may also be negative.

In any case, relations (6.12a) - (6.12b) show that the presence of nonlinear terms among the strain tensor components D_{ik} strengthens the asymmetry in the distribution of stresses over the thickness of the shell, in connection with the appearance of radial displacements w. In the zone of tension, the stresses increase more rapidly than would follow from the linear theory, while in the zone of compression they increase more slowly.

In this connection, the stresses in the tensile zone reach the yield point earlier than in the compressive zone, the material of the shell in the tensile zone loses its load-carrying capacity, and the active load is transmitted to the material in the initial zone of compression. Of course, in this case there is a redistribution of stresses, and the very concept of initial zone of compression loses its meaning. /205

To obtain a quantitative evaluation of the effect of nonlinearity on the deformation process, let us use one of the solutions of the problem in its linear postulation, given earlier (Bibl.23d).

Consider the case of the equilibrium of a tube freely resting on the face contours of the middle surface*. The function w(x) in this case is defined in the following manner **:

$$w = -\frac{4\nu T l^4}{\pi R}\sum_{n=0}^{\infty}\frac{\sin\frac{(2n+1)\pi x}{l}}{(2n+1)\left[D\pi^4(2n+1)^4 - T l^2\pi^2(2n+1)^2 + \beta l^4\right]}. \tag{6.13}$$

Here, R and l are, respectively, the radius and length of the tube, and D and β are expressed by the formulas

$$D = \frac{2Eh^3}{3(1-\nu^2)}, \quad \beta = \frac{2Eh}{R^2}, \tag{6.14a}$$

where E is Young's modulus, ν Poisson's constant, and E and ν are connected with the Lamé constants by the relations (II, 4.2a) - (II, 4.2b). The force compressing the tube is denoted by T. We shall now make a statement of substantial importance for what follows.

The parameter l in eq.(6.13) may also be understood as a quantity connected with the length of the tube by the relation

$$l_{mp} = kl, \tag{6.14b}$$

where k is a whole number. In this case, the function w(x) determined by eq.(6.13) will satisfy the boundary conditions and the fundamental differential

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* Freely resting is considered by us as equivalent to hingedly resting.

** Cf. [Bibl.23d, eq.(2.3)].

equation of the problem, since that equation does not contain the parameter l (Bibl.23d). Thus, we have the right to attribute to the parameter l a definite meaning predetermining the solution (6.13). We shall give this predetermination below.

The critical value of T [the upper critical value according to conventional terminology (Bibl.4, 10)] satisfies the equation

$$D\pi^4(2n+1)^4 - T\pi^2 l^2(2n+1)^2 + \beta l^4 = 0. \tag{6.15}$$

It can be established that the minimum critical value T corresponds to the following approximate relation:

$$l \approx (2n+1)\pi \sqrt[4]{\frac{D}{\beta}}. \tag{6.16}$$

Under condition (6.16), we find from eq.(6.15) the well-known formula /206

$$T_{cr} = \frac{4E}{\sqrt{3(1-\nu^2)}} \frac{h^2}{R}. \tag{6.17}$$

Using eq.(6.15) and relation (6.16), we find the following approximate expression for w(x):

$$w \cong -\frac{4\nu(Rh)^{\frac{1}{2}}}{[3(1-\nu^2)]^{\frac{3}{4}}} \frac{h}{l} \frac{1}{\frac{T_{cr}}{T}-1} \sin\frac{[3(1-\nu^2)]^{\frac{1}{4}}}{(Rh)^{\frac{1}{2}}} x. \tag{6.18}$$

The term retained here determines w(x) for a half-wave corresponding to the principal form of loss of stability. Therefore, l is here not the total length of the tube but the length of a half-wave. This length is indeterminate. For its determination we must use experimental data.

Making use of eq.(6.9), we find from eq.(6.18):

$$\overline{\Omega}_{31} = -\frac{4\nu}{[3(1-\nu^2)]^{\frac{1}{2}}} \frac{h}{l} \frac{1}{\frac{T_{cr}}{T}-1} \cos\frac{[3(1-\nu^2)]^{\frac{1}{4}}}{(Rh)^{\frac{1}{2}}} x. \tag{6.19a}$$

Putting $\nu = 0.3$, we obtain

$$\overline{\Omega}_{31} \cong -0{,}727 \frac{h}{l} \frac{1}{\frac{T_{cr}}{T}-1} \cos\frac{1{,}28}{(Rh)^{\frac{1}{2}}} x. \tag{6.19b}$$

We now also give expressions for $|\bar{\Omega}_{31}|_{max}$, $(\bar{\Omega}_{31})^2_{max}$, and $(\bar{\Omega}_{31})^4_{max}$:

$$|\bar{\Omega}_{31}|_{max} = 0.727 \, \frac{h}{l} \, \frac{1}{\frac{T_{cr}}{T} - 1}; \tag{6.20a}$$

$$(\bar{\Omega}_{31})^2_{max} = 0{,}523 \left(\frac{h}{l}\right)^2 \frac{1}{\left(\frac{T_{cr}}{T} - 1\right)^2}; \quad (\bar{\Omega}_{31})^4_{max} = 0{,}273 \left(\frac{h}{l}\right)^4 \frac{1}{\left(\frac{T_{cr}}{T} - 1\right)^4}. \tag{6.20b}$$

Equation (6.20a) determines the limits of variation of $\bar{\Omega}_{31}$. To determine $\bar{\lambda}^*$ and $\bar{\mu}^*$, according to eqs.(3.18a) - (3.18b), we have

$$\frac{1}{2b} \int_{-b}^{b} (\bar{\Omega}_{31})^2 \, d\bar{\Omega}_{31} = \frac{1}{3} (\bar{\Omega}_{31})^2_{max}; \quad \frac{1}{2b} \int_{-b}^{b} (\bar{\Omega}_{31})^4 \, d\bar{\Omega}_{31} = \frac{1}{5} (\bar{\Omega}_{31})^4_{max}. \tag{6.21}$$

where

$$b = |\bar{\Omega}_{13}|_{max}.$$

Returning again to eqs.(6.12a) - (6.12b) and noting on the basis of Table 1 that the principal significance here is possessed by the terms containing the component $\bar{\Omega}_{31}$, we find /207

$$\bar{\lambda}^* = \lambda + \lambda a^{-2} \left[\frac{1}{5} m_{10} (\bar{\Omega}_{31})^4_{max} + \frac{1}{3} m_{11} a (\bar{\Omega}_{31})^2_{max} + \right.$$
$$\left. + \frac{1}{3} m_{12} a^2 (\bar{\Omega}_{31})^2_{max} \right] + \mu a^{-2} \left[\frac{1}{5} n_{10} (\bar{\Omega}_{31})^4_{max} + \frac{1}{3} n_{11} a (\bar{\Omega}_{31})^2_{max} + \right.$$
$$\left. + \frac{1}{3} n_{12} a^2 (\bar{\Omega}_{31})^2_{max} \right]; \tag{6.22a}$$

$$\bar{\mu}^* = \mu + \lambda a^{-2} \left[\frac{1}{5} m_{20} (\bar{\Omega}_{31})^4_{max} + \frac{1}{3} m_{21} a (\bar{\Omega}_{31})^2_{max} + \right.$$
$$\left. + \frac{1}{3} m_{22} a^2 (\bar{\Omega}_{31})^2_{max} \right] + \mu a^{-2} \left[\frac{1}{5} n_{20} (\bar{\Omega}_{31})^4_{max} + \frac{1}{3} n_{21} a (\bar{\Omega}_{31})^2_{max} + \right.$$
$$\left. + \frac{1}{3} n_{22} a^2 (\bar{\Omega}_{31})^2_{max} \right]. \tag{6.22b}$$

These equations permit only a rather rough evaluation of the effects connected with the presence of nonlinear terms among the components of the finite-deformation tensor D_{ik}.

Consider now the principal conclusion resulting from the above.

1. Evaluation of the Effect of the Component Ω_{31} on the Stressed State of a Shell Depending on the Value of the Ratios $T_{cr} : T$ and $h : l$

It is clear from eqs.(6.20b), (6.21), and (6.22a) - (6.22b) that the effect of the nonlinear terms on the coefficients λ^* and μ^* substantially depends on the ratios $T_{cr} : T$ and $h : l$. Let us introduce the notation

$$\omega = \frac{h}{l}\left(\frac{T_{cr}}{T} - 1\right)^{-1}. \tag{6.23a}$$

Then,

$$|\bar{\Omega}_{31}|_{max} = 0,727\omega;\ (\bar{\Omega}_{31})^2_{max} = 0,523\omega^2;\ (\bar{\Omega}_{31})^4_{max} = 0,273\omega^4. \tag{6.23b}$$

As can be concluded from eqs.(6.22a) - (6.22b), the effect of the nonlinear terms among the components of D_{ik} decreases with decreasing ratio $h : l$ and increases as T approaches T_{cr}.

To disclose more distinctly the significance of the nonlinear terms, we present below certain numerical calculations of the value of the quantities entering into eqs.(6.22a) - (6.22b), based on the solution (6.18). Of course, the results so obtained can be regarded merely as rough and indicative. /208

Taking as before, $\nu = 0.3$ and, consequently, $\lambda = \frac{3}{2}\mu$, let us consider the separate terms in eqs.(6.22a) - (6.22b). Let us put, bearing in mind eqs.(6.23a) - (6.23b)

$$x_1 = 0,055a^{-2}\left(\frac{3}{2}m_{10} + n_{10}\right)\omega^4; \tag{6.24a}$$

$$x_2 = 0,174a^{-1}\left(\frac{3}{2}m_{11} + n_{11}\right)\omega^2; \tag{6.24b}$$

$$y_1 = 0,055a^{-2}\left(\frac{3}{2}m_{20} + n_{20}\right)\omega^4; \tag{6.24c}$$

$$y_2 = 0,174a^{-1}\left(\frac{3}{2}m_{21} + n_{21}\right)\omega^2. \tag{6.24d}$$

Then, eqs.(6.22a) - (6.22d) can be represented as follows:

$$\bar{\lambda}^* = \mu\left(\frac{3}{2} - |x_1| \pm x_2\right), \tag{6.25a}$$

$$\mu^* = \mu(1 + y_1 \pm y_2). \tag{6.25b}$$

In the relation (6.22a) - (6.22b), we neglected terms of relative order $(\bar{\Omega}_{31})^2_{max}$, retaining terms of orders $a^{-1}(\bar{\Omega}_{31})^2_{max}$, and $a^{-2}(\bar{\Omega}_{31})^4_{max}$ and took into consideration the values of the coefficients m_{ik} and n_{ik} given in Table 2. A positive sign before terms within parentheses corresponds, as above, to a region of tension.

It is clear from eqs.(6.25a) - (6.25b) and from Table 2 that the normal stresses in the zone of tension are greater than those determined by the linear theory, while in the zone of compression they are correspondingly less. It is clear that the shearing stresses in the zone of dominating tensile stresses, in areas inclined at an angle of $\frac{\pi}{4}$ to the generatrix, will be greater than those determined by the linear theory. To evaluate the degree of deviation of the nonlinear theory from the linear, let us turn to Tables 3 and 4. These Tables give the values of x_1 and y_1 and their corresponding values of ω.

It will be seen from Tables 3 and 4 and eqs.(6.25a) - (6.25b) that, even at small values of ω, the reduced elastic constants $\bar{\lambda}^*$ and $\bar{\mu}^*$ can deviate considerably from λ and μ, but this, according to eqs.(2.14), involves a deviation of the field of stresses from that found by the linear theory. /209

We call attention, for instance, to the figures marked by an asterisk (*) in Tables 3 and 4. Table 3 indicates that, for steel, at $\omega \approx 0.09$, in the zone of dominant tensile stresses $\bar{\lambda}^*$ is more than 30% greater than λ, whereas in the zone of dominant compressive stresses it is more than 35% smaller. For duralumin, these changes in $\bar{\lambda}^*$ already occur at $\omega \approx 0.07$.

TABLE 3

STEEL 30KHGSA				DURALUMIN D17			
x_1	ω	x_2	ω	x_1	ω	x_2	ω
0,05	0,0900*	0,05	0,0265	0,05	0,0664*	0,05	0,0195
0,10	0,107	0,10	0,0374	0,10	0,0790	0,10	0,0276
0,20	0,127	0,20	0,0530	0,20	0,0939	0.20	0,0391
0,30	0,141	0,30	0,0648	0,30	0,104	0,30	0,0478
0,40	0,151	0,40	0,0749	0,40	0,112	0,40	0,0552
0,50	0,160	0,50	0,0831*	0,50	0,118	0,50	0,0618*

TABLE 4

STEEL 30KHGSA				DURALUMIN D17			
y_1	ω	y_2	ω	y_1	ω	y_2	ω
0,05	0,0914	0,05	0,142*	0,05	0,0674	0,05	0,105*
0,10	0,109	0,10	0,201	0,10	0,0802	0,10	0,148
0,20	0,129*	0,20	0,284	0,20	0,0953	0,20	0,209
0,30	0,143*	0,30	0,348	0,30	0,106*	0,30	0,256
0,40	0,154	0,40	0,402	0,40	0,113	0,40	0,296
0,50	0,162	0,50	0,449	0,50	0,120	0,50	0,331

The reduced elastic constant $\bar{\mu}^*$ likewise differs appreciably from μ at relatively small values of ω. Thus, for example, it will be clear from Table 4 that the value $\bar{\mu}^*$ for steel, at $\omega \approx 0.14$, is about 35% greater than μ, in the zone of dominant tensile stresses, and 25% in the zone of compressive stresses. For duralumin, these changes of $\bar{\mu}^*$ occur already at $\omega \approx 0.11$.

We shall now determine whether the values of ω given above are possible. In the case under consideration, their existence is ensured by the factor $\left(\frac{T_{cr}}{T} - 1\right)^{-1}$ on the right-hand side of eq.(6.23a). Table 5 gives the values of ω, the ratios $\frac{h}{l}$ and the corresponding values of the ratio $T : T_{cr}$.

TABLE 5

ω	$h : l$	$T : T_{cr} = \omega : \left(\omega + \frac{h}{l}\right)$
	0,02	0,750
0,06	0,04	0,600
	0,06	0,500
	0,02	0,800
0,08	0,04	0,667
	0,06	0,571
	0,02	0,833
0,10	0,04	0,714
	0,06	0,625
	0,02	0,857
0,12	0,04	0,750
	0,06	0,667

Since the value of the ratio $h : l$ is unknown, let us turn to the ex- /210

perimental data given in the Vol'mir monograph (Bibl.4).

We relied on the solution of the axisymmetric problem. However, as shown by experiments, the shapes in which thin shells buckle are not axisymmetric. Two or three systems of depressions are formed, which can still be considered as a certain equivalent of the systems of half-waves of the axisymmetric form of deformation.

Using Figs.8.2 and 8.3 of the Vol'mir monograph (Bibl.4), we can evaluate the ratio l : R if we assume, as indicated above, that the parameter l is the length of a half-wave. This evaluation shows that the ratio l : R for specimens shown in Figs.8.2 and 8.3 (Bibl.4) can have values of 0.15 - 0.2.

Now, from eq.(6.16), for n = 0, i.e., for one half-wave, we find

$$\frac{h}{l}=\frac{\sqrt{3(1-\nu^2)}}{\pi^2}\,\frac{l}{R}. \tag{6.26a}$$

For ν = 0.3, we obtain

$$\frac{h}{l}=0{,}168\,\frac{l}{R}. \tag{6.26b}$$

Hence, we find that to the variation of the ratio l : R over the interval (0.15; 0.20) there corresponds a variation of the ratio h : l over the interval (0.025; 0.034). This calculation confirms the advisability of selecting the variational interval of the ratio h : l given in Table 5. Let us now find the ratio 2h : R. From the data presented here it follows that this ratio, for the specimens shown in Figs.8.2 and 8.3 of the Vol'mir monograph, varies over the interval $\left(\frac{1}{125},\frac{1}{70}\right)$. The determination of the ratio 2h : R is presented for verification. Judging from the content of the monograph (Bibl.4), the values of the obtained ratio 2h : R approximately correspond to the geometrical characteristics of the test specimens, since the ratio R : 2h= = 100 - 180 in the experiments described in that study. Of course, bearing eq.(6.26a), in mind, the analysis might be conducted without first having recourse to experimental data.

From eq.(6.26a), it follows that /211

$$\frac{l}{R}=\frac{\pi}{\sqrt[4]{3(1-\nu^2)}}\sqrt{\frac{h}{R}};\quad \frac{h}{l}=\frac{\sqrt[4]{3(1-\nu^2)}}{\pi}\sqrt{\frac{h}{R}}. \tag{6.26c}$$

Let, for example, 2h : R = 1 : 100. Then,

$$l:R\cong 0{,}165;\quad h:l\cong 0{,}029.$$

This corresponds to the variational interval of the ratios ℓ : R and h : ℓ found from the experiments given in the above monograph (Bibl.4).

To summarize we may say that our assignment of eqs.(6.13) to one half-wave is sufficiently motivated.

Returning to the question of the connection between the values of the reduced elastic constants $\bar{\lambda}^*$, $\bar{\mu}^*$ and the ratio T : T_{cr}, we find from Table 5 that the differences between $\bar{\lambda}^*$ and $\bar{\mu}^*$ and λ, μ, which reach 30% ± 5% in absolute value occur during the variation of the ratio T : T_{cr} over the interval (0.75; 0.85)*. In this case, the constants $\bar{\lambda}^*$ and $\bar{\mu}^*$ found for duralumin are more sensitive to the variations of the ratio T : T_{cr} than these same constants given for steel. To go deeper into the meaning of these conclusions, let us consider the simplified expression for w(x) differing from eq.(6.18), and let us draw several supplementary conclusions.

Let us return to eq.(6.13). If we again use eq.(6.15) to determine T_{cr} and hereafter take T_{cr} to mean its minimum value expressed by eq.(6.17), then from eq.(6.13) we can find:

$$w \cong -\frac{4\nu l^2}{\pi^3 R}\left(\frac{T_{cr}}{T}-1\right)^{-1}\sum_{n=0}^{\infty}\frac{\sin\frac{(2n+1)\pi x}{l}}{(2n+1)^3} \tag{6.27}$$

However**,

$$\sum_{n=0}^{\infty}\frac{\sin\frac{(2n+1)\pi x}{l}}{(2n+1)^3}=\frac{1}{8}\,\frac{\pi^3}{l^2}\,x(l-x) \qquad (0<x<l). \tag{6.28}$$

Consequently,

$$w \cong -\frac{\nu}{2R}\left(\frac{T_{cr}}{T}-1\right)^{-1}x(l-x). \tag{6.29}$$

It should be noted that eq.(6.29) is approximate, since instead of the roots of eq.(6.15), which are functions of n, we introduced into eq.(6.13) only the minimum value of the root, which was independent of n. Moreover, eq.(6.29) does not satisfy all the boundary conditions of the problem, although the expressions (6.27) obtained from this relation do satisfy the boundary conditions. This fact is connected with the well-known properties of the expansions of functions in Fourier series. /212

A direct comparison of the relative accuracy of eqs.(6.18 and (6.29) is

* This interval is stated as a rough approximation.

** Cf., for instance, L.V.Kantorovich, Definite Integrals and Fourier Series, Leningrad University, 1940

difficult and requires special investigation. Equation (6.29) has the advantage of simplicity over eq.(6.18). It also reflects the influence of the terms of the series that are rejected in deriving eq.(6.18).

Obviously a relatively small error in eqs.(6.18) and (6.29) may have a substantial effect on the results, since one must operate with the fourth and second powers of the component $\bar{\Omega}_{31}$.

Let us now discuss the physical meaning of eq.(6.29). Since the parameter n did not enter in eq.(6.29), the quantity l no longer has any definite meaning in this equation. Here, l may be taken to mean a segment, varying from the length of a half-wave determined by the sinusoid according to eq.(6.18), up to the entire length of the tube. We assume, as before, that l is the length of a half-wave.

The properties of Fourier series permit eq.(6.29) to be differentiated twice, and the resultant derivative will have meaning everywhere over the unclosed interval (0, l). Consequently, setting ν = 0.3, we find

$$|\bar{\Omega}_{31}|_{max}=\frac{\nu}{2}\frac{l}{R}\left(\frac{T_{cr}}{T}-1\right)^{-1}=0,15\Omega, \tag{6.30}$$

where

$$\Omega=\frac{l}{R}\left(\frac{T_{cr}}{T}-1\right)^{-1}, \tag{6.31}$$

and further

$$(\bar{\Omega}_{31})^2_{max}=0,023\Omega^2;\quad (\bar{\Omega}_{31})^4_{max}=0,00053\Omega^4. \tag{6.32}$$

Instead of eqs.(6.24a) - (6.24d), we obtain

$$\xi_1=0,0001a^{-2}\left(\frac{3}{2}m_{10}+n_{10}\right)\Omega^4, \tag{6.33a}$$

$$\xi_2=0,008a^{-1}\left(\frac{3}{2}m_{11}+n_{11}\right)\Omega^2, \tag{6.33b}$$

$$\eta_1=0,0001a^{-2}\left(\frac{3}{2}m_{20}+n_{20}\right)\Omega^4, \tag{6.33c}$$

$$\eta_2=0,0076a^{-1}\left(\frac{3}{2}m_{11}+n_{11}\right)\Omega^2. \tag{6.33d}$$

Equations (6.22a) - (6.22b) now take the following form: /213

$$\bar{\lambda}^* = \mu\left(\frac{3}{2} - |\xi_1| \pm \xi_2\right), \tag{6.34a}$$

$$\bar{\mu}^* = \mu\,(1 + \eta_1 \pm \eta_2). \tag{6.34b}$$

The data given in Tables 3 and 4 permit us to find Ω for a specified ξ_1 and η_1. For this, it is sufficient to multiply the values of ω by a "transition factor" equal to $0.727 : 0.15 \cong 4.85$. This factor can also be found for an arbitrary value of Poisson's constant ν, if we compare eqs.(6.19a) and (6.30). We have

$$\frac{h}{l} = \frac{\sqrt{3\,(1-\nu^2)}}{8}\,\frac{l}{R} \tag{b}$$

instead of eq.(6.26a).

A comparison of the "exact" relation (6.26a) with the approximate relation (b) shows that the use of eqs.(6.19a) and (6.30) involves a considerable error in the results, an error of the order of 20%. This undoubtedly is due to the fact that eqs.(6.19a) and (6.30) were obtained by differentiation of the approximate expressions $w(x)$.

If we use the same transition factor from ω to Ω and from $h : l$ to $l : R$, determined by eq.(6.26a) or relation (b), then the values of the ratio $T : T_{cr}$ will be independent of the choice of the transition factor, since in this case the following equation will be true:

$$T : T_{cr} = \omega : \left(\omega + \frac{h}{l}\right) = \Omega : \left(\Omega + \frac{l}{R}\right). \tag{c}$$

But since the accuracy of eqs.(6.29) - (6.30) may be greater than the accuracy of eqs.(6.18) - (6.19a), we shall consider the conclusions obtained from eqs.(6.30) - (6.34b) independently of the conclusions obtained from eq.(6.18) and its consequences. For this purpose, let us make use of Tables 3 and 4 with the transition factor (b) of 4.85 for $\nu = 0.3$ and consider the quantity $\frac{l}{R}$ as an independent parameter, which is equivalent to adopting the relation (6.26a) or (6.26b).

It is obvious that, under these conditions, the reduced constants $\bar{\lambda}^*$ and $\bar{\mu}^*$ will vary more rapidly as T approaches T_{cr} than they would according to the calculations based on eq.(6.18). In fact, at values of the ratio $l : R = 0.15 - 0.20$, the changes in the reduced constant $\bar{\mu}^*$ relative to μ, which in absolute value reach 30% ± 5%, occur while the ratio $T_{cr} : T$ varies over the interval (0.67, 0.79). In this case, $\bar{\lambda}^*$ varies more rapidly than $\bar{\mu}^*$.

Criterion of instability. We mentioned above that, in the zone of dominant compressive stresses, the parameter $\bar{\lambda}^*$ decreases and $\bar{\mu}^*$ increases. This points to a decrease in the reduced Poisson constant $\bar{\nu}^*$ in the zone of dominant compressive stresses. While decreasing, the constant $\bar{\nu}^*$ may become equal to zero and then become negative. Above, we did not consider the meaning of the change in sign of the reduced elastic constants. Let us fill this gap somewhat. /214

It is easy to prove that the specific work of deformation will have a positive definite quadratic form if the Poisson constant varies over the interval $\left(-1, \frac{1}{2}\right)$. Here, as is well known, the proof of the Kirchhoff theorem is valid and the solution of the boundary problems of the statics of linearly deformed bodies is unique, i.e., the state of equilibrium of such bodies is stable. However, no negative values of the Poisson constant have been found in actual isotropic bodies. Evidently, this experimental fact is not random but depends on the actual properties of matter which are not reflected by the simplified scheme of the continuous medium and, consequently, also not by the analytic structure of the specific energy of deformation.

Most authors assume that the Poisson constant is always positive. It was stated by E.Trefftz, erroneously, that the positive nature of Poisson's constant results from the requirement of the positive determinacy of the specific energy of deformation*. Let us assume, at first, according to experiments, that in a real isotropic body the Poisson constant is always positive and that the criterion of instability** is the change of sign of the reduced Poisson constant $\bar{\nu}^*$. Let us find the minimum value of the compressive force T at which the Poisson constant, in the zone of dominant compressive stresses, becomes negative. Equating $\bar{\lambda}^*$ to zero, we find, according to eq.(6.25),

$$|x_1| + x_2 - \frac{3}{2} = 0. \tag{6.35}$$

Making use of eqs.(6.24a) - (6.24b) and the data given in Table 2 for duralumin, we obtain an equation biquadratic in ω. The smallest positive real root of this equation is

$$\bar{\omega}_{cr} = 0{,}098. \tag{6.36}$$

It will be seen from Table 5 that to this value of ω_{cr} and to the ratio $h : \ell = 0.03$ corresponds the following value of the ratio $T : T_{cr}$: /215

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* E.Trefftz, Mathematical Theory of Elasticity ONTI, 1934, pp.39-40.

** The use of this instability criterion is not mandatory. A different and more natural approach is possible. See below, eqs.(6.63a) - (6.63b) etc..

$$T:T_{cr}=T_{\underline{cr}}:T_{cr}\cong 0{,}77. \tag{6.37}$$

where $T_{\underline{cr}}$ is the new critical value of the compressive force T. The second and greater root of the equation

$$|x_1|-x_2-\frac{3}{2}=0, \tag{6.38}$$

established for the zone of dominant tensile stresses, is

$$\bar{\omega}_{\underline{cr}}=0{,}246. \tag{6.39}$$

To this root, at the ratio $h : l = 0.03$, there corresponds a value of the ratio $T : T_{cr}$ close to unity. The first root corresponds to the arithmetic mean value of the critical load, and the second to the upper value. According to data given in the Vol'mir monograph (Bibl.4), the experimental ratio $T : T_{cr}$ ranges from 0.384 to 0.700.

Thus, the arithmetic mean of the critical load found by us approaches the upper limit of the experimental data.

Up to now we have been relying on eq.(6.35), derived from the approximate expression (6.18). Let us now make use of the equation

$$|\xi_1|+\xi_2-\frac{3}{2}=0, \tag{6.40}$$

resulting from eq.(6.34a). Obviously, the required solution of this equation will be obtained from the solution (6.36) after multiplying it by 4.85. As a result, we obtain

$$\bar{\Omega}_{\underline{cr}}=0{,}475. \tag{6.41}$$

Assuming that the ratio $l : R \cong 0.18$, we find that

$$T_{\underline{cr}}:T_{cr}\cong 0{,}74. \tag{6.42}$$

Consequently, here too the critical value of the load is outside the variational region of the experimental data.

Two possible causes of the contradiction between the results obtained by us for the approximate determination of the critical value of the load and the experimental data could be given.

The first is that we based our calculations on the averaged value of $(\bar{\Omega}_{\vartheta 1})_{max}$, instead of on $(\Omega_{\vartheta 1})_{max}$.

The second cause of the unsatisfactory results of the theoretical investigations undertaken by us lies in the choice of the parameter a. We have /216 already noted repeatedly that the proper selection of the region of approximations is of substantial importance. It is this choice that determines the value of the constant a, entering into the formulas determining λ^* and μ^*. We have assumed that the stresses in the shell reach the yield point, and from these conditions we have determined the value of a. But for sufficiently long and thin shells, the loss of stability may occur before the stresses reach the yield point. Therefore, the above conclusions are true only for sufficiently thick and short shells, in which there are no losses of stability under stresses approaching the yield point.

For other shells, one must decrease the values of the parameter a to below those shown in Table 1. This leads to an increased influence of the nonlinear terms. But the difficulty here consists in the determination of a.

Let us find the value of the ratio $\underline{T_{cr}} : T_{cr}$ starting from the unaveraged values of λ^* and μ^* corresponding to the study of local instability in a medium approximately equivalent to a shell.

Instead of eqs.(6.24a) - (6.24d) we will then have

$$\bar{x}_1 = 5x_1 = 0{,}273a^{-2}\left(\frac{3}{2}m_{10} + n_{10}\right)\omega^4, \tag{6.43a}$$

$$\bar{x}_2 = 3x_2 = 0{,}523a^{-1}\left(\frac{3}{2}m_{11} + n_{11}\right)\omega^2, \tag{6.43b}$$

$$\bar{y}_1 = 5y_1 = 0{,}273a^{-2}\left(\frac{3}{2}m_{20} + n_{20}\right)\omega^4, \tag{6.43c}$$

$$\bar{y}_2 = 3y_2 = 0{,}523a^{-1}\left(\frac{3}{2}m_{21} + n_{21}\right)\omega^2. \tag{6.43d}$$

Further, we find

$$\lambda^* = \mu\left(\frac{3}{2} - |\bar{x}_1| \pm \bar{x}_2\right), \tag{6.44a}$$

$$\mu^* = \mu(1 + \bar{y}_1 \pm \bar{y}_2). \tag{6.44b}$$

Equating λ^* to zero, we obtain an equation biquadratic in ω:

$$|\bar{x}_1| \mp \bar{x}_2 - \frac{3}{2} = 0. \tag{6.45}$$

The smallest positive real root of this equation, determined for duralumin, is as follows:

$$\underline{\omega}_{cr} = 0{,}058. \tag{6.46}$$

A direct calculation, together with Table 5, shows that this value of ω_{cr} and the ratio $h : l = 0.03$ correspond to /217

$$\underline{T}_{cr} : T_{cr} = 0{,}66. \tag{6.47}$$

According to the experiments described in the Vol'mir monograph (Bibl.4), the mean value of the ratio $T : T_{cr}$ is ~ 0.61*. The value of $T : T_{cr}$ found by us lies roughly at the center of the interval between the experimental mean value of $T : T_{cr}$ and the upper boundary of the experimental data, which is 0.7.

However, we derived eq.(6.47) by making use of the sequels of eq.(6.18). Let us consider the conclusions based on eq.(6.29). By analogy with eq.(6.41), we find

$$\underline{\Omega}_{cr} = 0{,}257. \tag{6.48}$$

Assuming as above that the ratio $l : R = 0.18$, we have

$$\underline{T}_{cr} : T_{cr} \cong 0{,}59. \tag{6.49}$$

This value is somewhat less than the mean experimental value given by Vol'mir (Bibl.4), which is 0.61. For 30KhGSA steel [eq.(6.48)], we get

$$\underline{\Omega}_{cr} = 0{,}384. \tag{6.50}$$

Accordingly,

$$\underline{T}_{cr} : T_{cr} \cong 0{,}66. \tag{6.51}$$

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* The value of Vol'mir's dimensionless parameter p (Bibl.4) corresponding to $T : T_{cr} = 0.66$ is 0.396. The mean value according to experiments is 0.364.

It will be clear from this that a calculation, using the maximum values of the components of Ω_{31}, yields satisfactory results.

Now let us consider the parameter a. As above, we shall confine ourselves to an approximate determination of this quantity.

We shall base our determination of the parameter a on eq.(6.29), which permits us to find the maximum bending stresses in the shell. This makes it possible to find the variational limits of the strain tensor components or the region of linear approximation of the components of the finite-deformation tensor or, in other words, the parameter a.

Double differentiation of the equivalent equations (6.27) and (6.29) is permissible, since the trigonometric series obtained on differentiation of eq.(6.27) converges to the derivative of the right-hand side of eq.(6.29).

Starting from eq.(6.29), we find from our paper (Bibl.23d) the maximum absolute value of the normal stress due to flexure and compression:

$$|\sigma_{max}| = \frac{T}{2h} + \frac{E\nu}{1-\nu^2}\frac{h}{R}\left(\frac{T_{cr}}{T} - 1\right)^{-1}. \tag{6.52}$$

Hence, we find the new value of the parameter a. Let us call this value a*. Making use of eq.(6.17), we find /218

$$a^* = \frac{|\sigma_{max}|}{E} = \frac{2}{\sqrt{3(1-\nu^2)}}\frac{h}{R}\frac{T}{T_{cr}} + \frac{\nu}{1-\nu^2}\frac{h}{R}\left(\frac{T_{cr}}{T} - 1\right)^{-1}. \tag{6.53a}$$

Equation (6.53a) can be put into the following form:

$$a^* = \left[\frac{2}{\sqrt{3(1-\nu^2)}}\frac{1}{\Omega + \frac{l}{R}} + \frac{\nu}{1-\nu^2}\frac{R}{l}\right]\frac{h}{R}\Omega. \tag{6.53b}$$

Further, bearing in mind eqs.(6.30), (6.32) and considering the unaveraged values of λ^* and μ^*, we find, by analogy to eqs.(6.33a) - (6.33b), the relations

$$\bar{\xi}_1 = 0{,}0005\left[\frac{2}{\sqrt{3(1-\nu^2)}}\frac{2}{\Omega + \frac{l}{R}} + \frac{\nu}{1-\nu^2}\frac{R}{l}\right]^{-2}\left(\frac{h}{R}\right)^{-2} \times$$

$$\times\left(\frac{3}{2}m_{10} + n_{10}\right)\Omega^2, \tag{6.54a}$$

$$\bar{\xi}_2 = 0{,}023\left[\frac{2}{\sqrt{3(1-\nu^2)}}\,\frac{1}{\Omega+\frac{l}{R}}+\frac{\nu}{1-\nu^2}\,\frac{R}{l}\right]^{-1}\left(\frac{h}{R}\right)^{-1}\times$$

$$\times\left(\frac{3}{2}\,m_{11}+n_{11}\right)\Omega. \tag{6.54b}$$

It goes without saying that these relations are true only for $\nu = 0.3$, since we have already assumed that $\lambda = \frac{3}{2}\mu$. We retained here the literal designation ν only to make the transformation more distinct. We have further

$$\lambda^* = \mu\left(\frac{3}{2} - |\bar{\xi}_1| \pm \bar{\xi}_2\right). \tag{6.55}$$

The equation for determining the critical values of Ω here takes the following form:

$$|\bar{\xi}_1| + \bar{\xi}_2 - \frac{3}{2} = 0. \tag{6.56}$$

This is an equation of the fourth degree in Ω. Let us denote its smallest positive root by Ω_{cr}. We find, without solving eq.(6.56), the approximate value of Ω_{cr}, making use of the previous calculations.

Now, comparing eqs.(6.54a) - (6.54b) with eqs.(6.33a) - (6.33b) and /219 bearing in mind that the difference in the numerical factors is connected with the fact that eqs.(6.54a) - (6.54b) yield the values of λ^* and μ^*, corresponding to the maximum values of the components of $\Omega_{\beta 1}$, instead of the mean values, $\bar{\lambda}^*$ and $\bar{\mu}^*$, we get

$$\bar{\Omega}_{cr} = \left[\frac{2}{\sqrt{3(1-\nu^2)}}\,\frac{1}{\bar{\Omega}_{cr}+\frac{l}{R}}+\frac{\nu}{1-\nu^2}\,\frac{R}{l}\right]\frac{h}{R}\,a^{-1}\,\Omega^2_{cr}. \tag{6.57}$$

This equation can be put into the following form:

$$\bar{\Omega}^2_{cr} + 2p\sqrt{\frac{h}{R}}\,\bar{\Omega}_{cr} - q\,\frac{h}{R} = 0. \tag{6.58}$$

where

$$p = \frac{\sqrt[4]{3(1-\nu^2)}}{2\pi}\left[\frac{\pi^2}{\sqrt{3(1-\nu^2)}} - \frac{\nu}{1-\nu^2}\,a^{-1}\,\bar{\Omega}^2_{\underline{cr}}\right] > 0, \tag{6.59a}$$

$$q = \left[\frac{2}{\sqrt{3(1-\nu^2)}} + \frac{\nu}{1-\nu^2}\right] a^{-1}\,\bar{\Omega}^2_{\underline{cr}}, \tag{6.59b}$$

Consequently, the required value of $\bar{\Omega}_{\underline{cr}}$ is

$$\bar{\Omega}_{\underline{cr}} = \left(\sqrt{p^2+q} - p\right)\sqrt{\frac{h}{R}} \tag{6.60}$$

while the corresponding value of the ratio T : T_{cr} is

$$T_{\underline{cr}} : T_{cr} = \frac{\bar{\Omega}_{\underline{cr}}}{\bar{\Omega}_{\underline{cr}} + \frac{l}{R}} = \frac{\sqrt{p^2+q} - p}{\sqrt{p^2+q} - p + \frac{\pi}{\sqrt[4]{3(1-\nu^2)}}}. \tag{6.61}$$

Thus, the ratio $T_{\underline{cr}}$: T_{cr} does not depend explicitly on the ratio h : R. The implicit dependence of the ratio $T_{\underline{cr}}$: T_{cr} on h : R is connected with the presence of the quantity $\Omega^2_{\underline{cr}}$ in eq.(6.58). This result is apparently confirmed by the experimentally established fact of the weak dependence of the ratio $T_{\underline{cr}}$: T_{cr} on the ratio h : R*. Equation (6.61) does not confirm in an explicit form the established experimental tendency of the ratio $T_{\underline{cr}}$: T_{cr} to decrease with decreasing ratio $\frac{h}{R}$**.

Consider now the numerical value of the ratio $T_{\underline{cr}}$: T_{cr} for duralumin, putting $\nu = 0.3$; $a^{-1} = 2.28 \times 10^2$; $\bar{\Omega}_{\underline{cr}} = 0.257$ [the value of $\bar{\Omega}_{\underline{cr}}$ is taken from eq.(6.48)]. We have /220

$$T_{\underline{cr}} : T_{cr} \cong 0{,}654. \tag{6.62}$$

Consequently, the methods of determining the parameter a increase the deviation of the ratio $T_{\underline{cr}}$: T_{cr} from the mean experimental value, as compared with the previous method, which led to eq.(6.49).

We note also that eqs.(6.57) - (6.58) can be used at small values of the ratio h : R, i.e., for very thin shells, since already at 2h : R = 0.01 the normal stress $|\sigma_{max}|$ calculated from eq.(6.52) exceeds the yield point for duralumin D17.

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* Cf. (Bibl.4, p.322).

** Ibid.

We have used various methods, based on the general method of linearization, in investigating the values of the ratio $T_{\underline{cr}} : T_{cr}$ corresponding to the vanishing of the Poisson constant ν^* of a linearly deformable medium, approximately equivalent to a medium with finite deformations.

Despite the difference in the analytic expressions of the radial displacement w used by us and despite the employment of both averaged and maximum absolute values for the component Ω_{31}, the values of the ratio $T_{\underline{cr}} : T_{cr}$ were found to be rather stable under variations of their determination methods. This shows that the conclusions obtained here cannot be due to random agreement of the numerical results. Rather, this must be a reflection of actual processes taking place on any loss of stability of the shell.

Let us now return to determination of the ratio $T_{\underline{cr}} : T_{cr}$. Above, we have used the change in sign of ν^* or $\bar{\nu}^*$ only as a criterion of instability.

A different approach to determination of the ratio $T_{\underline{cr}} : T_{cr}$ is possible, based on the use of eq.(6.17), in which the elastic constants E and ν must be substituted by E^* and ν^*, corresponding to $|\Omega_{31}|_{max}$, or $\bar{E}^*$ and $\bar{\nu}^*$ found from the average values of $|\Omega_{31}|$. Using eqs.(6.17) and (6.23a), we obtain

$$T_{\underline{cr}} : T_{cr} = \frac{\omega}{\omega + \frac{h}{l}} = \frac{E^*}{E}\sqrt{\frac{1-\nu^2}{1-\nu^{*2}}} \qquad (6.63a)$$

and, on using the averaged values of $|\Omega_{31}|$,

$$T_{\underline{cr}} : T_{cr} = \frac{\omega}{\omega + \frac{h}{l}} = \frac{\bar{E}^*}{E}\sqrt{\frac{1-\nu^2}{1-\bar{\nu}^{*2}}}. \qquad (6.63b)$$

If we express E^* and ν^* in terms of λ^* and μ^*, and use eqs.(6.44a)-(6.44b) then eq.(6.63a) is transformed into an equation algebraic in ω. A similar /221
equation can be obtained from eq.(6.63b).

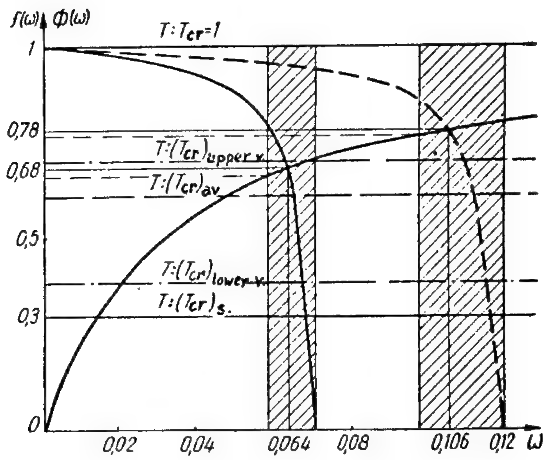
To find the smallest real positive roots of these equations, corresponding to the minimum values of the ratio $T_{\underline{cr}} : T_{cr}$, we employed a graphical method. Let us put

$$f(\omega) = \frac{\omega}{\omega + \frac{h}{l}}; \quad \Phi(\omega) = \frac{E^*}{E}\sqrt{\frac{1-\nu^2}{1-\nu^{*2}}};$$

$$\bar{\Phi}(\omega) = \frac{\bar{E}^*}{E}\sqrt{\frac{1-\nu^2}{1-\bar{\nu}^{2*}}}, \qquad (6.64)$$

assuming $\nu = 0.3$; $\frac{h}{l} \cong 0.03$.

Let us introduce the coordinates $(\omega, f\Phi)$ (see graph).



Plotting on the plane $(\omega, f\Phi)$ or $(\omega, \overline{f\Phi})$ the functions $f(\omega)$, $\Phi(\omega)$ and $\overline{\Phi}(\omega)$, we find the roots sought. It will be seen from the graph that the value of $\underline{T_{cr}} : T_{cr}$ found from eq.(6.63a) is 0.68, while that calculated from eq.(6.63b) is 0.78.

The graph also shows that the values of ν^* and $\overline{\nu^*}$, corresponding to the critical values of the load, lie in the interval (-1, 0). For ν^* equal to -1, the quantity E^* vanishes, which evidently corresponds to a complete loss of the load-carrying capacity of the shell. At $\overline{\nu^*} = -1$, $\overline{E^*}$ similarly vanishes.

Thus the region of instability corresponds to the variation of ν^* and $\overline{\nu^*}$ over the interval (-1, 0). These regions are shown by hatching in the graph.

We note further that the points of the planes $(\omega, f\Phi)$ or $(\omega, f\overline{\Phi})$ cor-, /222 responding to the critical values of the ratio $T : T_{cr}$ always lie in the instability regions between the theoretical upper values and the theoretical lower value of this ratio.

If we use eq.(6.63a), then the value of $\underline{T_{cr}} : T_{cr} = 0.68$ found by us is very close to the arithmetic mean of the upper and lower values of this ratio found by the theoretical energetic method. The arithmetic mean of the upper and lower critical values of the ratio $T : T_{cr}$ is 0.65. All this confirms the expedience of using the method we have considered in the problems of stability of a shell. This method permits an approximate determination of the mean values of the critical load if we find its upper value from the solution of the problem in linear formulation. It is clear that the same method makes it also possible to evaluate the lower critical value of the load. For example, finding that $\underline{T_{cr}} : T_{cr}$ equals 0.68, we determine the lower critical value of this ratio as $\overline{0.68} - (100 - 0.68) = 0.36$. In the Vol'mir monograph (Bibl.4) it is shown that the lower critical value of the ratio $T : T_{cr}$ is about 0.33.

We shall not consider the results of the application of eq.(6.29) although, as will be seen from the preceding discussion, we can here obtain a

value of the ratio T : T_{cr} still closer to the mean theoretical and experimental value.

Section 7. Brief Conclusions

The contents of the preceding Section confirm the usefulness of applying the method of linearization of the components of the finite-deformation tensor developed by us at the beginning of this Chapter. The difference between this method and other approximate methods of solving nonlinear problems, for example the method of Galerkin and Ritz, is that we did not start out from the content of special problems suggesting the form of the approximation function, but tried to derive the general principles of construction of the approximate solution suitable for extensive classes of boundary problems. Strictly speaking, our method is suitable for the approximate solution of any nonlinear boundary problem of shell theory. This method, in its concept, is close to the well-known methods of analytic synthesis of mechanisms according to P.L.Chebyshev and was therefore included by us in the principles of the analytic mechanics of shells.

We shall make concluding remarks on the correlation between the consequences of the method of linear approximation of the components of the finite-deformation tensor and the theory of stability of shells.

1. On the Mechanism of Development of a Local Equilibrium and Motion of Instability of a Shell /223

A study of the properties of a continuous medium with small deformations, approximately replacing a shell with finite deformations, yields a preliminary idea as to the course of the development of a local equilibrium or motion instability of a shell.

When the load approaches the upper critical value, there is an increase of the asymmetry of stress distribution over the thickness of the shell.

In the zone of dominant tensile stresses, the yield point is reached considerably before the load increases to the upper critical limit. After loss of the load-carrying capacity, by the material in the zone of tension, the load is transmitted to the material in the zone of dominant compressive stresses, if the material in that zone is in the stable state.

On the basis of the properties of the medium approximately equivalent to the material of the shell, it can only be said that, in the zone of compressive stresses, processes take place that approximate the state of this substituted medium to the unstable state. This is characterized by a change in sign of Poisson's constant ν^* and its passage through zero.

The instability of the medium is accompanied by a motion of its elements, representing the elements of the shell, in accordance with the Gauss principle of least constraint, or with the Le Chatelier-Brown principle*.

*The Le Chatelier-Brown principle is as follows: If any stress or force is brought to bear upon a system in equilibrium, the equilibrium is (cont'd)

These principles are in particular directly connected with the methods of studying the theory of stability of shells recently proposed by the author A.V.Pogorelov*.

2. The Role of Random Imperfections of Shape

The losses of stability of a shell constitute a nonstationary dynamic wave process which originally arises as a result of various random sources, even if the load is far from the upper critical value and there are none of the flexural deformations considered by us in the preceding Subsection**. /224
These random factors include the initial imperfections of shape, which may be interpreted as the existence of initial finite displacements of points of the middle surface.

To this initial displacement correspond components of the antisymmetric tensor Ω_{ik}. Returning to the above axisymmetric problem, let us assume that the deviation from the cylindrical shape of the basic surface is determined by the function $w_0(x)$. To this function corresponds a finite component $(\bar{\Omega}_{31})_0$. As will be seen from eqs.(6.22a) - (6.22b), the existence of the component $(\bar{\Omega}_{31})_0$ leads to a greater variation of the field of stresses under loading of the shell even if the terms $\frac{1}{5}\left(\frac{3}{2}\ m_{10} + n_{10}\right)(\bar{\Omega}_{31})_0^4$ and $\frac{1}{3}\left(\frac{3}{2}\,m_{11} + n_{11}\right)(\bar{\Omega}_{31})_0^2$ (i = 1, 2) are of the order of a^2, i.e., very small in magnitude.

These quantities may exert a great influence on $\bar{\lambda}^*$ and λ^* and, consequently, may decrease the value of the critical load below that found above.

3. Regions of Static Instability

If we make use of eq.(6.19a) and substitute Ω_{31} into eqs.(6.12a) - (6.12b), then λ^* and μ^* will be periodic functions of the x-coordinate. To obtain the next approximation, let us use eqs.(2.14), to find the stress tensor components. Then, applying one of the methods of reduction, we shall obtain the system of equations of equilibrium of a cylindrical shell, but this system of linear equations, as already noted in Sect.5, will have variable coefficients - in this case periodic coefficients - which depend on the parameter $\left(\frac{T_{cr}}{T} - 1\right)^{-1}$

(cont'd) displaced in a direction which tends to diminish the intensity of the stress or force. (Cf. L.Landau and Ye.Lifshits, Statistical Physics, Gostekhizdat, 1940).

* Cf. A.V.Pogorelov, Contribution to the Theory of Elastic Shells in the Transcritical Stage, Kharkov University, 1960. The isometric deformations of the middle surface of the shell on which his theory is based are directly connected with these principles.

** Various aspects of this idea may be found in the monographs of V.V.Bolotin (Bibl.2b, 2c).

Such equations show the possibility of the existence of instability regions replacing the isolated critical values of T obtained from the quasilinear equations of the first approximation. These same remarks apply to the problems of dynamics.

A complete investigation of the questions touched upon here would be beyond the scope of this book. We note again that a number of the above properties for nonlinearly deformed shells can be found by other methods, without the use of the linear approximation developed by us for the components of the finite-deformation tensor.

Section 8. Construction of a Homogeneous Isotropic Shell Approximately Equivalent to a Layered Shell /225

In Sects. 26-27 of Chapter III, we considered the equations of motion of a two-layered shell. Retaining twelve degrees of freedom on the normal to the basic surface of the shell, we obtained a system of equations of motion of the thirty-sixth order. Clearly, we must seek methods for obtaining a mathematical formulation of the problem that would make it solvable in practice.

We shall here consider the method of solving the problem of the motion of a layered shell, based on the approximate replacement of this shell by a homogeneous shell. Studies in this direction, and a study of a simplified system of equations by means of the selection of a basic surface of the layered shell, have been performed by E.L.Aksel'rad, E.L.Grigolyuk, and V.I.Korolev (Bibl.15a, b, 21, 24).

In contrast to these investigators, we shall here apply methods of approximation functions connected with the requirement of the least-square error in constructing the Lagrange function L* of a homogeneous shell approximately equivalent of the layered shell.

We shall here indicate three methods of solving this problem. The first is based on the consideration of an incompatible system of algebraic equations established independently of the properties of the variables entering into the Lagrange function. The second method is connected with a general evaluation of the magnitudes of these variables. The third method relies on a preliminary solution of specific problems of the dynamics of homogeneous shells. We shall not base our work here on (III, 26.8 - 26.11), since the presence of the covariant derivatives $\nabla_i u_j^{(*)}$ in the expressions for the coefficients $V_j^{(0)}$ and $V_j^{(1)}$ makes these equations unsuitable for solution of the problem with which we are now concerned.

We will make use of an idea which is the inverse of that advanced by L.A.Molotkov in one of his papers on elastic waves in layered media*. He considers a medium, inhomogeneous in the direction of one of the coordinates, as the limiting case of a layered medium. We shall consider the layered shell as

* L.A.Molotkov. Engineering Equations of Vibrations of Plates with a Layered Structure. Questions of the Dynamic Theory of the Propagation of Seismic Waves, Vol. V, Leningrad otd. Inst. matem. AN SSSR, 1961

a special case of a shell inhomogeneous in the direction of the normal to its basic surface. This method of studying the mechanics of layered shells of course involves the difficulties which will be discussed below.

1. Application of an Incompatible System of Algebraic Equations /226

We shall confine ourselves to a study of the question in its linear formulation. Consider a shell inhomogeneous in the direction of the normal to the basic surface. Let us select the basic surface as indicated in (III), Sect.25). The object of the approximation will be the Lagrange function LdS of a prismatic element of a shell of height 2h with the base area dS:

$$L = T - \Pi. \tag{8.1}$$

where TdS and ΠdS are, respectively, the kinetic energy and the strain energy of this element of the shell.

The functions L, T and Π are the respective densities of the Lagrange function, of the kinetic energy, and of the strain energy.

Hereafter, for brevity, we shall often omit the term "density" and call L, T and Π respectively the Lagrange function, the kinetic energy and the strain energy.

Let us construct a homogeneous shell of thickness $2h^*$, of density ρ^* and with the elastic Lamé constants λ^* and μ^*, assuming that the basic surface of the inhomogeneous shell and the basic surface of the homogeneous shell, approximately equivalent to it, coincide, starting out from the condition of least error

$$\Delta = L - L^*, \tag{8.2}$$

which arises in the above-indicated substitution.

The required quantities ρ^*, λ^*, μ^* and h^* must be found from the conditions of optimum approximate representation* of the function L by the function L^*.

The number of available quantities increases to six if we abandon the preliminary selection of the basic surfaces in the layered shell and the shell equivalent to it. The idea of selecting the basic surface so as to introduce simplifications into the system of equations of the theory of layered shells is discussed elsewhere (Bibl.15, 24).

Since we intend in the following to discuss only the principles of the

* The meaning of the requirement of "optimum representation" will be explained below.

proposed method, we will almost everywhere confine ourselves to an arbitrary selection of the four above parameters.

We shall confine ourselves to the problem of determining these quantities only as functions ρ_i, λ_i, μ_i, h_i for the layered shell, dispensing with the study of their connection with the metric of the shell. In that case, we may pass to the local Cartesian system of rectangular coordinates with the axis OZ directed along the normal to the basic surface inside the shell, and the axes Ox^i located in a plane tangent to the basic surface. In this case, $g_{ii} = g^{ii} = 1$, $g^{ik} = 0$ $(i \neq k)$: an element of volume is expressed by the derivative $dx^1 dx^2 dz$, and the element of area dS will be equal to $dx^1 dx^2$.

In considering Δ we must express L and L* in the same variables. In the choice of these variables we shall start out from the well-known properties of the fields of displacements, strains, and stresses in a layered shell. /227

It has been shown in Chapter III, Sects.25-27, that, in a layered shell, the components of the displacement vector, the components ϵ_{ik} (i, k = 1, 2) of the strain tensor, and the components τ_{i3} (i = 1, 2) of the stress tensor are continuous.

We shall base our work on the assumption that the fields of these quantities coincide in the layered shell and the equivalent homogeneous shell to within the limits of the prismatic element mentioned above.

It is well known that all continuous functions of the coordinate z can be approximated by polynomials of z. Since we shall determine L and L* with an accuracy to terms of the order h^3 and h^{*3}, and intend to give here only the general principles of the method proposed, let us put

$$\tau_{j3} \cong \tau_{j3}^{(0)} + z\tau_{j3}^{(1)} + \frac{1}{2} z^2 \tau_{j3}^{(2)} \qquad (j = 1,\ 2,\ 3), \tag{8.3}$$

$$\theta = \frac{\partial u_1}{\partial x^1} + \frac{\partial u_2}{\partial x^2} = \theta^{(0)} + z\theta^{(1)} + \frac{1}{2} z^2 \theta^{(2)}. \tag{8.4}$$

Further, from Hooke's law, we find

$$u_3 = u_3^{(0)} + \frac{1}{\lambda + 2\mu} \sum_{m=0}^{2} \frac{z^{m+1}}{(m+1)!} [\tau_{33}^{(m)} - \lambda\theta^{(m)}]. \tag{8.5}$$

Making use of eqs.(8.3) and (8.5), we obtain

$$u_i = u_i^{(0)} + \sum_{m=0}^{2} \frac{z^{m+1}}{(m+1)!} \left[\frac{\tau_{i3}^{(m)}}{\mu} - u_{3,i}^{(m)} \right] \qquad (i = 1,\ 2). \tag{8.6}$$

where

$$u_{3,i}^{(0)} = \partial_i u_3^{(0)}; \quad u_{3,i}^{(m)} = \frac{1}{\lambda+2\mu}\,\partial_i [\tau_{33}^{(m-1)} - \lambda\theta^{(m-1)}] =$$

$$= \frac{1}{\lambda+2\mu}[\tau_{33,i}^{(m-1)} - \lambda\theta_{,i}^{(m-1)}] \qquad (i,\ m=1,\ 2). \tag{8.7}$$

Equations (8.5) - (8.6) show that the representation of the components of the displacements by (III, 15.5) is inapplicable to the problems of the vibrations of a layered shell.

In fact, as will be clear from eqs.(8.5) - (8.6), the continuous coefficients of z^m in the formulas determining the components u_j are expressed in terms of piecewise-continuous functions of z, constant on those segments of the OZ axis included within the layers. This dependence of the coefficients /228 of z^m on z is not reflected in explicit form by (III, 15.5).

Let us also approximate the continuous components ε_{ik} of the strain tensor by the polynomials

$$\varepsilon_{ik} = \varepsilon_{ik}^{(0)} + z\varepsilon_{ik}^{(1)} + \frac{1}{2}z^2\varepsilon_{ik}^{(2)} \qquad (i,\ k=1,\ 2), \tag{8.8a}$$

in this connection

$$\theta^{(m)} = \sum_{i=1}^{2}\varepsilon_{ii}^{(m)}. \tag{8.8b}$$

Now let us express the discontinuous components of the strain tensor and the stress tensor in terms of continuous components. We have

$$\varepsilon_{i3} = \frac{1}{2\mu}\sum_{m=0}^{2}\frac{1}{m!}z^m\tau_{i3}^{(m)}, \tag{8.9a}$$

$$\varepsilon_{33} = \frac{1}{\lambda+2\mu}\sum_{m=0}^{2}\frac{z^m}{m!}[\tau_{33}^{(m)} - \lambda\theta^{(m)}]; \tag{8.9b}$$

$$\tau_{ii} = \sum_{m=0}^{2}\frac{z^m}{m!}\left\{\frac{\lambda}{\lambda+2\mu}[\tau_{33}^{(m)} + 2\mu\theta^{(m)}] + 2\mu\varepsilon_{ii}^{(m)}\right\}; \tag{8.10a}$$

$$\tau_{ik} = 2\mu\sum_{m=0}^{2}\frac{z^m}{m!}\varepsilon_{ik}^{(m)} \qquad (i,\ k=1,\ 2). \tag{8.10b}$$

Since we are investigating the local properties of the function L, without going outside the boundaries of the prismatic element of the shell defined above, we must consider the derivatives ∂_i $(i = 1, 2)$ of the coefficients of the polynomials here introduced as new, locally independent, quantities. This, more particularly, explains the application of the independent representations, by the polynomials, of the components ϵ_{ik} without inversion to eqs.(8.5)-(8.6), since this would not lead to a decrease in the number of locally independent quantities introduced by us.

All the locally independent quantities entering into eqs.(8.3) - (8.10b) belong to the variable field in terms of which the Lagrange function L of an element of the continuous medium is expressed. They are a generalization of the generalized coordinates and generalized velocities known from classical mechanics*. We recall that in the classical Lagrange function, the generalized velocities q_j and the generalized coordinates q_j are considered as inde- /229
pendent quantities in setting up the equations of motion. Their interrelation is taken into account after setting up the equations of motion based on the elementary equations $q_j = \frac{dq_j}{dt}$. In the problem of interest to us, the interrelation of the variable fields is expressed by more complex relations resulting from the equations of elasticity theory. For example, from eqs.(8.4) and (8.6) we may find

$$\theta^{(0)} = \sum_{i=1}^{2} \partial_i u_i^{(0)}; \quad \theta^{(m)} = \sum_{i=1}^{2} \partial_i \left[\frac{\tau_{i3}^{(m-1)}}{\mu} - u_{3,i}^{(m-1)} \right]. \tag{8.11}$$

Equation (6.7) must be associated with these relations.

A number of relations result from eqs.(8.6) - (8.10b), but we shall not consider them here, since we are not setting up a system of equations of motion of a layered shell by the methods of classical analytical mechanics, but propose to make use of the Lagrange function L as the fundamental quantity in the problem of constructing a homogeneous shell approximately equivalent to a layered shell.

It follows from eqs.(8.3) - (8.10b) that the number of variable fields entering into the Lagrange function is thirty-one. It is easy to establish that this number does not depend on the number of layers n in the shell if $n > 1$. For $n = 1$, the number of variables of the field can be reduced to twenty-seven.

Consider now the kinetic energy T related to unit area of the basic surface. In the local system of rectangular Cartesian coordinates, we have

$$2T = \int_0^{2h} \rho \sum_{j=1}^{3} (\dot{u}_j)^2 \, dz. \tag{8.12}$$

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* Cf., for example, J.Leach, Classical Mechanics, Chapter IX, IL, 1961.

Equation (8.12) is applicable to an inhomogeneous shell. Let us pass to a layered shell, putting:

$$\rho=\rho_1+\sum_{k=1}^{n-1}\Delta\rho_k\sigma_0(z-z_k), \tag{8.13a}$$

$$\lambda=\lambda_1+\sum_{k=1}^{n-1}\Delta\lambda_k\sigma_0(z-z_k), \tag{8.13b}$$

$$\mu=\mu_1+\sum_{k=1}^{n-1}\Delta\mu_k\sigma_0(z-z_k), \tag{8.13c}$$

where $\Delta\rho_k$, $\Delta\lambda_k$, $\Delta\mu_k$ are the changes in ρ, λ, μ on transition from the k^{th} layer to the $(k+1)^{th}$ layer, n is the number of layers, σ_0 is the unit Heaviside function, and z_k are the coordinates of the interfaces of the layers. It is assumed that the layers are parallel, i.e., that the coordinates z_k are constants. /230

To shorten the formulas we shall also make use of relations of the form of eqs.(8.13a) - (8.13c) in considering the functions ρ, λ, μ.

For example,

$$f(\rho,\lambda,\mu)=f(\rho_1,\lambda_1,\mu_1)+\sum_{k=1}^{n-1}\Delta f(\rho_k,\lambda_k,\mu_k)\sigma_0(z-z_k). \tag{8.14}$$

Let us also introduce the notation:

$$A_m(f)=\int_0^{2h}f(\rho,\lambda,\mu)z^m dz=\frac{(2h)^{m+1}}{m+1}f(\rho_1,\lambda_1,\mu_1)+$$

$$+\sum_{k=1}^{n-1}\frac{(2h)^{m+1}-z_k^{m+1}}{m+1}\Delta f(\rho_k,\lambda_k,\mu_k) \tag{8.15}$$

and retain in eqs.(8.5) - (8.6) only the terms containing the factors z^0 to z^2 inclusive. This makes it possible to find T approximately to terms with factors of the order of h^3. In this case, T will contain only seventeen variable fields. We have

$$(\dot{u}_i)^2=(\dot{u}_i^{(0)})^2+2z\dot{u}_i^{(0)}\left[\frac{\dot{\tau}_{i3}^{(0)}}{\mu}-\dot{u}_{3,i}^{(0)}\right]+z^2\left\{\dot{u}_i^{(0)}\left[\frac{\dot{\tau}_{i3}^{(1)}}{\mu}-\dot{u}_{3,i}^{(1)}\right]+\right.$$

$$+\left[\frac{\dot{\tau}_{i3}^{(0)}}{\mu}-\dot{u}_{3,i}^{(0)}\right]^2\Bigg\}\qquad (i=1,\ 2);$$

(8.16a)

$$(\dot{u}_3)^2=(\dot{u}_3^{(0)})^2+\frac{2z}{\lambda+2\mu}\,\dot{u}_3^{(0)}[\dot{\tau}_{33}^{(0)}-\lambda\dot{\theta}^{(0)}]+z^2\Bigg\{\frac{\dot{u}_3^{(0)}}{\lambda+2\mu}[\dot{\tau}_{33}^{(1)}-\lambda\dot{\theta}^{(1)}]+$$

$$+\frac{1}{(\lambda+2\mu)^2}[\dot{\tau}_{33}^{(0)}-\lambda\dot{\theta}^{(0)}]^2\Bigg\}. \qquad (8.16b)$$

Substituting these equations into eq.(8.12), and making use of eqs.(8.7), (8.14), (8.15), we find

$$2T=A_0(\rho)\sum_{j=1}^{3}(\dot{u}_j^{(0)})^2+\sum_{i=1}^{2}\Bigg\{2A_1\left(\frac{\rho}{\mu}\right)\dot{u}_i^{(0)}\dot{\tau}_{i3}^{(0)}-2A_1(\rho)\,\dot{u}_i^{(0)}\dot{u}_{3,i}^{(0)}+$$

$$+A_2\left(\frac{\rho}{\mu}\right)[\dot{u}_i^{(0)}\dot{\tau}_{i3}^{(1)}-2\dot{\tau}_{i3}^{(0)}\dot{u}_{3,i}^{(0)}]-A_2\left(\frac{\rho}{\lambda+2\mu}\right)\dot{u}_i^{(0)}\dot{\tau}_{33,i}^{(0)}+$$ /231

$$+A_2\left(\frac{\rho\lambda}{\lambda+2\mu}\right)\dot{u}_i^{(0)}\dot{\theta}_{,i}^{(0)}+A_2\left(\frac{\rho}{\mu^2}\right)(\dot{\tau}_{i3}^{(0)})^2+A_2(\rho)\,(\dot{u}_{3,i}^{(0)})^2\Bigg\}+$$

$$+2A_1\left(\frac{\rho}{\lambda+2\mu}\right)\dot{u}_3^{(0)}\dot{\tau}_{33}^{(0)}-2A_1\left(\frac{\rho\lambda}{\lambda+2\mu}\right)\dot{u}_3^{(0)}\dot{\theta}^{(0)}+$$

$$+A_2\left(\frac{\rho}{\lambda+2\mu}\right)\dot{u}_3^{(0)}\dot{\tau}_{33}^{(1)}-A_2\left(\frac{\rho\lambda}{\lambda+2\mu}\right)\dot{u}_3^{(0)}\dot{\theta}^{(1)}+A_2\left(\frac{\rho}{(\lambda+2\mu)^2}\right)\times$$

$$\times(\dot{\tau}_{33}^{(0)})^2+A_2\left(\frac{\rho\lambda^2}{(\lambda+2\mu)^2}\right)(\dot{\theta}^{(0)})^2-2A_2\left(\frac{\rho\lambda}{(\lambda+2\mu)^2}\right)\dot{\tau}_{33}^{(0)}\dot{\theta}^{(0)}.\ (8.17^a)$$

(8.17a)

Equation (8.17a) can be put into a somewhat different form:

$$2T=A_0(\rho)\sum_{j=1}^{3}(\dot{u}_j^{(0)})^2-2A_1(\rho)\sum_{i=1}^{2}\dot{u}_i^{(0)}\dot{u}_{3,i}^{(0)}+A_2(\rho)\sum_{i=1}^{2}(\dot{u}_{3,i}^{(0)})^2+$$

$$+2A_1\left(\frac{\rho}{\mu}\right)\sum_{i=1}^{2}\dot{u}_i^{(0)}\dot{\tau}_{i3}^{(0)}+A_2\left(\frac{\rho}{\mu}\right)\sum_{i=1}^{2}[\dot{u}_i^{(0)}\dot{\tau}_{i3}^{(1)}-2\dot{\tau}_{i3}^{(0)}\dot{u}_{3,i}^{(0)}]+$$

$$+2A_1\left(\frac{\rho}{\lambda+2\mu}\right)\dot{u}_3^{(0)}\dot{\tau}_{33}^{(0)}+A_2\left(\frac{\rho}{\lambda+2\mu}\right)\left[\dot{u}_3^{(0)}\dot{\tau}_{33}^{(1)}-\sum_{i=1}^{2}\dot{u}_i^{(0)}\dot{\tau}_{33,i}^{(0)}\right]-$$

$$-2A_1\left(\frac{\rho\lambda}{\lambda+2\mu}\right)\dot{u}_3^{(0)}\dot{\theta}^{(0)}+A_2\left(\frac{\rho\lambda}{\lambda+2\mu}\right)\left[\sum_{i=1}^{2}\dot{u}_i^{(0)}\dot{\theta}_{,i}^{(0)}-\dot{u}_3^{(0)}\dot{\theta}^{(1)}\right]+$$

$$+A_2\left(\frac{\rho}{\mu^2}\right)\sum_{i=1}^{2}(\dot{\tau}_{i3}^{(0)})^2+A_2\left(\frac{\rho}{(\lambda+2\mu)^2}\right)(\dot{\tau}_{33}^{(0)})^2+A_2\left(\frac{\rho\lambda^2}{(\lambda+2\mu)^2}\right)\times$$

$$\times(\dot{\theta}^{(0)})^2-2A_2\left(\frac{\rho\lambda}{(\lambda+2\mu)^2}\right)\dot{\tau}_{33}^{(0)}\dot{\theta}^{(0)}. \qquad (8.17b)$$

Further, we have

$$2\Pi=\int_0^{2h}[\tau_{11}\varepsilon_{11}+\tau_{22}\varepsilon_{22}+\tau_{33}\varepsilon_{33}+2(\tau_{12}\varepsilon_{12}+\tau_{13}\varepsilon_{13}+\tau_{23}\varepsilon_{23})]\,dz. \qquad (8.18)$$

Confining ourselves to the relative accuracy adopted in calculating T, and making use of eqs.(8.8a) - (8.10b), we find

$$2\Pi=A_0\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)(\theta^{(0)})^2+2A_1\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)\theta^{(0)}\theta^{(1)}+A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)\times$$

$$\times[\theta^{(0)}\theta^{(2)}+(\theta^{(1)})^2]+A_0(2\mu)\left[\sum_{i=1}^{2}(\varepsilon_{ii}^{(0)})^2+2(\varepsilon_{12}^{(0)})^2\right]+2A_1(2\mu)\times$$

$$\times\left[\sum_{i=1}^{2}\varepsilon_{ii}^{(0)}\varepsilon_{ii}^{(1)}+2\varepsilon_{12}^{(0)}\varepsilon_{12}^{(1)}\right]+A_2(2\mu)\left\{\sum_{i=1}^{2}[\varepsilon_{ii}^{(0)}\varepsilon_{ii}^{(2)}+(\varepsilon_{ii}^{(1)})^2]+\right.$$

$$\left.+2[\varepsilon_{12}^{(0)}\varepsilon_{12}^{(2)}+(\varepsilon_{12}^{(1)})^2]\right\}+A_0\left(\frac{1}{\lambda+2\mu}\right)(\tau_{33}^{(0)})^2+2A_1\left(\frac{1}{\lambda+2\mu}\right)\times$$

$$\times\tau_{33}^{(0)}\tau_{33}^{(1)}+A_2\left(\frac{1}{\lambda+2\mu}\right)[\tau_{33}^{(0)}\tau_{33}^{(2)}+(\tau_{33}^{(1)})^2]+2A_0\left(\frac{1}{2\mu}\right)\sum_{i=1}^{2}(\tau_{i3}^{(0)})^2+$$

$$+4A_1\left(\frac{1}{2\mu}\right)\sum_{i=1}^{2}\tau_{i3}^{(0)}\tau_{i3}^{(1)}+2A_2\left(\frac{1}{2\mu}\right)\sum_{i=1}^{2}[\tau_{i3}^{(0)}\tau_{i3}^{(2)}+(\tau_{i3}^{(1)})^2]. \qquad (8.19)$$

Here the quantities $\theta^{(m)}$ and $\varepsilon_{ii}^{(m)}$ are connected by the relation (8.8b).

Equations (8.17) and (8.19) complete the construction of the function L for a layered shell with the accuracy adopted by us. As a special case, these equations yield the expression of the Lagrange function L* for a homogeneous (single-layer) shell in the variable fields selected by us.

For a single-layer shell, eq.(8.15) takes the following form:

$$A_m^*(f^*) = \frac{(2h^*)^{m+1}}{m+1} f^*. \tag{8.20}$$

Using this notation, we can obtain 2T* and 2Π* directly from eqs.(8.17a)-(8.17b) and (8.19), but since all this reduces down to substituting the operators $A_m^*(f^*)$ for $A_m(f)$, we shall not write out the expressions for 2T* and 2Π*.

Let us return to the problem of the approximation of the Lagrange function L by the function L*.

Let us consider the difference Δ. As will be seen from eqs.(6.2) and the properties of T, Π, T*, Π*, this difference in turn is a Lagrange function with the coefficients

$$\Delta_m = A_m(f) - A_m^*(f^*). \tag{8.21}$$

If a function L* existed equal to L, then all the differences Δ_m, and thus also the difference Δ, would vanish. This vanishing of Δ would mean the existence of a single-layer shell equivalent in this respect to a multi-layer shell. But we have available only four quantities characterizing the properties of the single-layer shell: ρ*, λ*, μ* and h*. Equating all the Δ_m to zero, we obtain, as will be seen from eqs.(6.17b) and (6.19), a system of twenty-five equations in four unknowns: /233

$$\Delta_m = A_m(f) - A_m^*(f^*) = 0. \tag{8.22}$$

The system (8.22) is incompatible. Consequently, it is impossible to construct a homogeneous shell equivalent to a multi-layer shell*.

We can speak only of approximate equivalents.

It is well known that there are several methods of constructing solutions

* This conclusion was obvious in advance, since, if the functions L and L* were exactly equal, and the external loads and boundary conditions of a single-layer shell coincided with that of a multi-layer shell, the single-layer shell would exactly imitate the motion of the multi-layer shell.

approximately satisfying a system of incompatible equations.

Let us apply the method based on the requirement of minimizing the sum of the squares of Δ_m. Let us consider the sum in the expanded form, making use of eqs.(8.17b), (8.19) and (8.20):

$$S_0=\sum\Delta_m^2=[A_0(\rho)-2h^*\rho^*]^2+\left[A_1(\rho)-\frac{1}{2}(2h^*)^2\rho^*\right]^2+$$

$$+\left[A_2(\rho)-\frac{1}{3}(2h^*)^3\rho^*\right]^2+\left[A_1\left(\frac{\rho}{\mu}\right)-\frac{1}{2}(2h^*)^2\frac{\rho^*}{\mu^*}\right]^2+$$

$$+\left[A_2\left(\frac{\rho}{\mu}\right)-\frac{1}{3}(2h^*)^3\frac{\rho^*}{\mu^*}\right]^2+\left[A_1\left(\frac{\rho}{\lambda+2\mu}\right)-\frac{1}{2}(2h^*)^2\times\right.$$

$$\left.\times\frac{\rho^*}{\lambda^*+2\mu^*}\right]^2+\left[A_2\left(\frac{\rho}{\lambda+2\mu}\right)-\frac{1}{3}(2h^*)^3\frac{\rho^*}{\lambda^*+2\mu^*}\right]^2+$$

$$+\left[A_1\left(\frac{\rho\lambda}{\lambda+2\mu}\right)-\frac{1}{2}(2h^*)^2\frac{\rho^*\lambda^*}{\lambda^*+2\mu^*}\right]^2+\left[A_2\left(\frac{\rho\lambda}{\lambda+2\mu}\right)-\right.$$

$$\left.-\frac{1}{3}(2h^*)^3\frac{\rho^*\lambda^*}{\lambda^*+2\mu^*}\right]^2+\left[A_2\left(\frac{\rho}{\mu^2}\right)-\frac{1}{3}(2h^*)^3\frac{\rho^*}{\mu^{*2}}\right]^2+$$

$$+\left[A_2\left(\frac{\rho}{(\lambda+2\mu)^2}\right)-\frac{1}{3}(2h^*)^3\frac{\rho^*}{(\lambda^*+2\mu^*)^2}\right]^2+\left[A_2\left(\frac{\rho\lambda^2}{(\lambda+2\mu)^2}\right)-\right.$$

$$\left.-\frac{1}{3}(2h^*)^3\frac{\rho^*\lambda^{*2}}{(\lambda^*+2\mu^*)^2}\right]^2+\left[A_2\left(\frac{\rho\lambda}{(\lambda+2\mu)^2}\right)-\frac{1}{3}(2h^*)^3\frac{\rho^*\lambda^*}{(\lambda^*+2\mu^*)^2}\right]^2+$$

$$+\left[A_0\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)-(2h^*)\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]^2+\left[A_1\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)-\frac{1}{2}(2h^*)^2\times\right.$$

$$\left.\times\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]^2+\left[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)-\frac{1}{3}(2h^*)^3\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]^2+$$

$$+[A_0(2\mu)-(2h^*)2\mu^*]^2+\left[A_1(2\mu)-\frac{1}{2}(2h^*)^2 2\mu^*\right]^2+$$

$$+\left[A_2(2\mu)-\frac{1}{3}(2h^*)^3 2\mu^*\right]^2+\left[A_0\left(\frac{1}{\lambda+2\mu}\right)-2h^*\frac{1}{\lambda^*+2\mu^*}\right]^2+$$

$$+\left[A_1\left(\frac{1}{\lambda+2\mu}\right)-\frac{1}{2}(2h^*)^2\frac{1}{\lambda^*+2\mu^*}\right]^2+\left[A_2\left(\frac{1}{\lambda+2\mu}\right)-\right.$$

$$\left.-\frac{1}{3}(2h^*)^3\frac{1}{\lambda^*+2\mu^*}\right]^2+\left[A_0\left(\frac{1}{2\mu}\right)-(2h^*)\frac{1}{2\mu^*}\right]^2+\left[A_1\left(\frac{1}{2\mu}\right)-\right.$$

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$$\left.-\frac{1}{2}(2h^*)^2\frac{1}{2\mu^*}\right]^2+\left[A_2\left(\frac{1}{2\mu}\right)-\frac{1}{3}(2h^*)^3\frac{1}{2\mu^*}\right]^2. \tag{8.23}$$

Let us introduce the notation

$$2h^*\rho^*=m^*;\quad 2h^*\mu^*=G^*;\quad 2h^*\lambda^*=H^*. \tag{8.24}$$

To determine the unknown parameters characterizing the homogeneous shell, let us set up the equations:

$$\frac{\partial S_0}{\partial m^*}=0;\quad \frac{\partial S_0}{\partial G^*}=0;\quad \frac{\partial S_0}{\partial H^*}=0;\quad \frac{\partial S_0}{\partial h^*}=0. \tag{8.25}$$

From the system of nonlinear equations (8.25) we determine the parameters m^*, G^*, H^* and h^*, after which the construction of the homogeneous shell, approximately replacing the layered shell, will be completed.

Several remarks must be made on the method proposed here for the construction of an equivalent homogeneous shell.

a. The system of nonlinear equations (8.25) for sufficiently small $\Delta\rho_1$, $\Delta\lambda_1$ and $\Delta\mu_1$, obviously has at least one system of real solutions, since for $\Delta\rho_1$, $\Delta\lambda_1$ and $\Delta\mu_1$ equal to zero, we obtain the solution $\rho^* = \rho_1$, $\lambda^* = \lambda_1$, $\mu^* = \mu_1$, $2h^* = h_1 = 2h$.

b) The incompatible system of equations (8.22) and the related function S_0 were considered by us apart from the difference $L - L^*$. We certainly had the right to proceed in this way, but the solution proposed involves the implicit assumption that those functions of the variable field in the expression for $L - L^*$ whose coefficients are the differences Δ_m on the left-hand sides of the incompatible equations (8.22) all have the same physical significance. This is undoubtedly the vulnerable point of the method. Evidently, instead of the function S_0, we should consider the function of a more general type:

$$S_1=\Sigma c_m\Delta_m^2, \tag{8.26}$$

where c_m is the weight of the term Δ_m^2 characterizing the physical significance of the corresponding term* in the expression for $L - L^*$.

The principal difficulty here lies in the determination of the numbers c_m. We shall first give an elementary example of the choice of the coefficients c_m, based on the classical theory of shells. /235

The Kirchhoff-Love hypotheses lead to the conclusion that, in the Lagrange functions L and L^*, we may neglect all terms containing variable fields connected with the components ϵ_{i2} of the strain tensor and ϵ_{i3} of the stress tensor. Consequently, in eq.(8.26) we should equate the coefficients c_m of the corresponding Δ_m to zero. If we put the remaining coefficients c_m as equal to unity, then we find

$$
\begin{aligned}
S_{10} = {} & [A_0(\rho) - (2h^*)\rho^*]^2 + 4\left[A_1(\rho) - \frac{1}{2}(2h^*)^2\rho^*\right]^2 + \\
& + \left[A_2(\rho) - \frac{1}{3}(2h^*)^3\rho^*\right]^2 + 4\left[A_1\left(\frac{\rho\lambda}{\lambda+2\mu}\right) - \frac{1}{2}(2h^*)^2 \times \right. \\
& \left. \times \frac{\rho^*\lambda^*}{\lambda^*+2\mu^*}\right]^2 + \left[A_2\left(\frac{\rho\lambda}{\lambda+2\mu}\right) - \frac{1}{3}(2h^*)^3\frac{\rho^*\lambda^*}{\lambda^*+2\mu^*}\right]^2 + \\
& + \left[A_2\left(\frac{\rho\lambda^2}{(\lambda+2\mu)^2}\right) - \frac{1}{3}(2h^*)^3\frac{\rho^*\lambda^{*2}}{(\lambda^*+2\mu^*)^2}\right]^2 + \left[A_0\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - (2h^*) \times \right. \\
& \left. \times \frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]^2 + 4\left[A_1\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \frac{1}{2}(2h^*)^2\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]^2 + \\
& + \left[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \frac{1}{3}(2h^*)^3\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]^2 + [A_0(2\mu) - (2h^*)2\mu^*]^2 + \\
& + 4\left[A_1(2\mu) - \frac{1}{2}(2h^*)^2 2\mu^*\right]^2 + \left[A_2(2\mu) - \frac{1}{3}(2h^*)^3 2\mu^*\right]^2.
\end{aligned}
\tag{8.27}
$$

In contrast to eq.(8.23), we introduced the coefficient 2^2 in certain terms of eq.(8.27), bearing in mind the numerical coefficients of the corresponding terms in the expressions for $2T$ and 2Π. This again corresponds to the assumption of the same physical significance of the functions of the variable fields for all coefficients of the form $aA_m(f)$ where a in this case equals unity or two.

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* Cf., for example, V.Ya.Goncharov, Theory of Interpolation and Approximation Functions, ONTI, 1934, p.161

We now consider the case when the components $\epsilon_{ik}^{(0)}(i,k = 1,2)$ vanish on deformation of the shell. In this case, we cannot neglect the terms containing the components $\tau_{i3}^{(m)}$ of the stress tensor.

The determination of the required parameters, characterizing the properties of a homogeneous shell approximately equivalent to a layered shell, is again reduced to the solution of the system of equations:

$$\frac{\partial S_{10}}{\partial m^*}=0;\quad \frac{\partial S_{10}}{\partial G^*}=0;\quad \frac{\partial S_{10}}{\partial H^*}=0;\quad \frac{\partial S_{10}}{\partial h^*}=0. \tag{8.28}$$

The methods based on the consideration of the sums S_0 and S_{10} defined by eq.(8.27) yield the result of averaging incompatible values of the required unknowns obtained from the system of equations (8.22). /236

This result of averaging, as will be seen from the above discussion, permits constructing of an equivalent shell with rough approximation, since the physical meaning of the quantities entering into T and Π, which are factors of Δ_m, differ substantially. We must therefore consider in greater detail the coefficients c_i of eq.(8.26).

2. Evaluation of the Weights c_i

We admit that the technique proposed below for evaluating the weights c_i in eq.(8.26) is quite imperfect. However, it permits an introduction, into the calculations, of quantities approximately characterizing the significance and physical properties of various groups of terms entering into the function L - L*. Here, as before, it is necessary to refine the region of variation of the variable field, since this region is at the same time the region of approximation of the function L by the function L*.

We shall assume that the shell undergoes stationary vibrations at a frequency ω lying in the interval (ω_1, ω_2). The quantities ω_1 and ω_2 are assumed to be known. To determine the frequency ω_1 we may use any approximation method, for example the Ritz method. The upper value of the frequency ω may be selected arbitrarily. We will show the influence of this choice. Let us assume, for instance, that the displacements u_i are expressed by the equations

$$u_i=U_i\begin{cases}\sin\omega t\\ \cos\omega t\end{cases}. \tag{8.29}$$

Let us substitute eq.(8.29) into eqs.(8.17b) and (8.19) and then average the results over the two-dimensional region $\left(\omega_1, \omega_2; 0, \frac{2\pi}{\omega}\right)$. Let us bear in mind the equations

$$\int_{\omega_1}^{\omega_2}\int_0^{\frac{2\pi}{\omega}} \cos^2 \omega t \, dt \, d\omega = \int_{\omega_1}^{\omega_2}\int_0^{\frac{2\pi}{\omega}} \sin^2 \omega t \, dt \, d\omega = \frac{1}{2}\int_{\omega_1}^{\omega_2}\int_0^{\frac{2\pi}{\omega}} dt \, d\omega = \pi \ln \frac{\omega_2}{\omega_1}; \tag{a}$$

$$\int_0^{\frac{2\pi}{\omega}} \cos \omega t \sin \omega t \, dt = 0; \tag{b}$$

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$$\int_{\omega_1}^{\omega_2} \omega^2 \int_0^{\frac{2\pi}{\omega}} \cos^2 \omega t \, dt \, d\omega = \int_{\omega_1}^{\omega_2} \omega^2 \int_0^{\frac{2\pi}{\omega}} \sin^2 \omega t \, dt \, d\omega = \frac{\pi}{2} (\omega_2^2 - \omega_1^2). \tag{c}$$

Let us temporarily introduce into the consideration the variable Ψ_f, defined by the relation

$$\psi_f = \frac{\partial_i f}{f}. \tag{d}$$

The variables Ψ_f are defined by the values taken by the ratio $(\partial_i f) : (f)$ on the basic surface, where f is the general symbol for the functions characterizing the stress-strain state of the shell. Obviously the variables Ψ_f are in particular connected with the variational indices of the function f (Bibl.5)

In considering specific problems on the vibrations of layered shells, we can sometimes determine in advance the approximate limits c_1 and c_2 within which the values of the variables Ψ_f will lie, from the known solutions of similar problems for homogeneous shells, and then average over Ψ_f the difference L - L* on the intervals (c_1, c_2).

As a result, the constants $c_{(m)}$, defined by the equation

$$c_{(m)} = \frac{1}{c_2 - c_1} \int_{c_1}^{c_2} (\psi_f)^m \, d\psi_f. \tag{e}$$

enter the equations.

Since the quantities $c_{(m)}$ have a definite meaning only for very narrow classes of problems, we shall below apply various methods permitting their exclusion from the equations solved.

Let us continue our study of the variational integrals of the variable fields entering into the Lagrange functions of homogeneous and layered shells, assuming that the components u_j $(j = 1, 2, 3)$ of displacement vary over the intervals (-2h, +2h). It is well known that, if this interval is further extended, the linear theory becomes unsuitable.

The greatest difficulties are connected with indicating the variational intervals of the quantities $\epsilon_{ik}^{(m)}$ and $\tau_{i3}^{(m)}$ (i, k = 1, 2; m = 0, 1, 2). These intervals depend largely on the special properties of the solutions of definite classes of boundary problems. We shall start out from the hypotheses that yield results in the general form, admitting of further simplification and connected with the special properties of specific problems.

Assume that the components $\epsilon_{ik}^{(0)}$ (i, k = 1, 2) of the strain tensor vary over the interval (-a, +a) where a is the greatest value taken by this quantity (cf. Sect.2) in the various layers. The exceptions are the cases in which we know in advance that the $\epsilon_{ik}^{(0)}$ are small or zero, when that range of variation contracts to a point. /238

Assume further that the components $\tau_{i3}^{(0)}$ of the stress tensor vary over the interval $(-c_\sigma, +c_\sigma)$. If there is reason to suppose that the stressed state of the shell is almost momentless, then the quantity c_σ must be determined from the relation

$$c_\sigma = 2\sigma_s k_{max} \tag{8.30a}$$

in accordance with the interval of variation $\epsilon_{ik}^{(0)}$. Here σ_s is the maximum yield point of the materials of the layers and k_{max} the greatest value of the principal curvature of the basic surface.

In purely flexural deformations of the shell, taking place in the absence of loads on its boundary surfaces, the absolute values of the components $\tau_{i3}^{(m)}$ (i = 1, 2, 3; m = 0, 1, 2) are small. In this case, to obtain approximate solutions, the interval $(-c_\sigma, c_\sigma)$ must be contracted to a point.

Assume further that all the nonzero quantities $\epsilon_{ij}^{(m)}$ (i, j = 1, 2; m = 1,2) vary over the range $[-(-2h)^m a, (2h)^m a]$ while the quantities $\tau_{i3}^{(m)}$ (i = 1, 2, 3; m = 1, 2) vary over the interval from $[-(2h)^m c_\sigma, (2h)^m c_\sigma]$. This is equivalent to the assumption that the terms actually entering into the approximation polynomials of the form (8.3) are of the same relative order.

All the above compels the conclusion that weighted quadratic approximations should be introduced after sufficiently complete concretization of the content of the problem of shell mechanics.

Let us turn now to a consideration of the special case of the determining of the elastic constants of the homogeneous shell, approximately equivalent to a layered shell.

3. Application of the Weighted Quadratic Approximation

Let us form the sum S_1, assuming that among the terms of the difference L - L* there are no zeroes and that an averaging has been made over the above intervals of variation of the variable fields and over their time derivatives entering into the composition of the kinetic energy.

In considering the terms in 2T, which contain derivatives with respect to the coordinates x^i, with a comma in the indices, let us use the variable Ψ_f according to eq.(d), followed by averaging. Thus the final result will contain the constants $c_{(s)}$, defined by eq.(e).

Let us introduce the notation /239

$$\Delta\omega = \frac{\omega_2^2 - \omega_1^2}{2(\ln\omega_2 - \ln\omega_1)}. \tag{8.30b}$$

Now, based on eqs.(8.17b) and (8.19), let us form the sum S_1 of the squares of the deviations from zero of the averaged values of the independent summands entering into the difference L - L*.

As before, let us consider the variable fields introduced by us and their derivatives as independent variables. We have

$$\begin{aligned}
S_1 = \sum c_m\Delta_m^2 = (\Delta\omega)^2 \Bigg\{ & 16h^4 [A_0(\rho) - 2h^*\rho^*]^2 + \frac{64}{9} h^4 c_{(2)} \Bigg[A_2(\rho) - \\
& - \frac{1}{3}(2h^*)^3 \rho^* \Bigg]^2 + \frac{4c_\sigma^4}{9} \Bigg[A_2\left(\frac{\rho}{\mu^2}\right) - \frac{1}{3}(2h^*)^3 \frac{\rho^*}{\mu^{*2}} \Bigg]^2 + \frac{c_\sigma^4}{9} \times \\
& \times \Bigg[A_2\left(\frac{\rho}{(\lambda+2\mu)^2}\right) - \frac{1}{3}(2h^*)^3 \frac{\rho^*}{(\lambda^*+2\mu^*)^2} \Bigg]^2 + \frac{4}{9} a^4 \Bigg[A_2\left(\frac{\rho\lambda^2}{(\lambda+2\mu)^2}\right) - \\
& - \frac{1}{3}(2h^*)^3 \frac{\rho^*\lambda^{*2}}{(\lambda^*+2\mu^*)^2} \Bigg]^2 \Bigg\} + \frac{4}{9} a^4 \Bigg[A_0\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - (2h^*) \times \\
& \times \frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*} \Bigg]^2 + \frac{64}{9} h^{-4} a^4 \Bigg[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \frac{1}{3}(2h^*)^3 \frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*} \Bigg]^2 + \\
& + a^4 [A_0(2\mu) - (2h^*)(2\mu^*)]^2 + 16h^{-4} a^4 \Bigg[A_2(2\mu) - \frac{1}{3}(2h^*)^3 2\mu^* \Bigg]^2 + \\
& + \frac{c_\sigma^4}{9} \Bigg[A_0\left(\frac{1}{\lambda+2\mu}\right) - (2h^*) \frac{1}{\lambda^*+2\mu^*} \Bigg]^2 + \frac{16}{9} h^{-4} c_\sigma^4 \Bigg[A_2\left(\frac{1}{\lambda+2\mu}\right) - \\
& - \frac{1}{3}(2h^*)^3 \frac{1}{\lambda^*+2\mu^*} \Bigg]^2 + \frac{16}{9} c_\sigma^4 \Bigg[A_0\left(\frac{1}{2\mu}\right) - (2h^*) \frac{1}{2\mu^*} \Bigg]^2 + \\
& + \frac{64}{9} h^{-4} c_\sigma^4 \Bigg[A_2\left(\frac{1}{2\mu}\right) - \frac{1}{3}(2h^*)^3 \frac{1}{2\mu^*} \Bigg]^2.
\end{aligned} \tag{8.31}$$

The following short remarks apply to the sum S_1, supplementing the statements in Subsection 2 of this Section.

a) The sum S_1 contains a smaller number of summands than the sum S_0, since, on averaging over symmetric intervals, the terms containing odd powers of the variable fields will cancel out.

b) As already noted, a term containing a factor $c_{(2)}$, can receive a definite meaning only for distinctly restricted classes of dynamic problems.

In static problems it will not be necessary to investigate the variables Ψ_f, and the constant $c_{(2)}$ will not enter into the equations. In the remaining cases an attempt must be made to eliminate the terms with the factor $c_{(2)}$. The most general method of eliminating one of the terms with such a factor is to choose the basic surface in the equivalent homogeneous shell such that the coefficient will vanish if the factor $c_{(2)}$ occurs in the selected term. /240

Equation (8.31) contains only one term with the factor $c_{(2)}$. Let us denote by h_1 the z coordinate determining the new position of the basic surface relative to the above-selected surface. To determine h_1, let us set up the following equation resulting from eq.(8.31):

$$[(2h^* - h_1)^3 + h_1^3]\,\rho^* = 3A_2(\rho). \tag{8.32}$$

If eq.(8.31) still contains another term with the factor $c_{(2)}$, the position of the basic surface would also have to be changed in the layered shell.

c) Introduction of the frequencies ω_1 and ω_2 shows that the approximate replacement of the layered shell by a homogeneous shell permits investigation of only a limited region of the frequency spectrum. The higher frequencies cannot be determined by this method.

d) Equation (8.31) was obtained under certain assumptions, which might fail to correspond to the physical content of individual problems. But the form of the relation (8.31) permits its adaptation to a number of special cases. This has already been mentioned above.

To supplement the above we note that in the case of plates, the coefficient c_σ must be taken as zero, and for flat shells close to zero, in accordance with eqs.(8.30a). These cases approximate the assumptions under which the sum S_{10} was obtained.

If a plate, under certain boundary conditions, has no chain stresses, then the evaluation of the variational interval of the stress tensor components τ_{13} by means of the quantity c_σ defined by the difference (8.30a) loses its meaning. In these cases, one must start out from the special conditions of loading of the boundary surfaces of the shell. At considerable surface densities of the load, c_σ may be taken as equal to σ_s, i.e., we may average the stress tensor components τ_{13} over their natural interval. Of course, this applies also to the corresponding cases of deformation of shells.

We emphasize in conclusion that the expression obtained by us for the

sum S_1 should be regarded merely as an example of the application of the general method. In solving special problems one must strive toward a preliminary individualization of the field variables, permitting different variational intervals to be prescribed for them and permitting their relative magnitude to be estimated, as noted above. For this reason, the formal application of eq.(8.31) to arbitrary problems cannot be recommended. /241

e) Equation (8.31), as was assumed, approximately reflects the physical significance of the individual terms entering into the sum S_1.

Of course in a comparative estimate of the magnitudes of the individual summands of the sum S_1 one must, as already noted, start from the physical content of the problems of a definite class. This permits an introduction of further simplifications into eq.(8.31). Let us consider one version of these simplifications, starting out from the basic assumptions made in deriving eq.(8.31).

There, we assumed that the tangential components of the stress tensor and the strain tensor differ from zero and that the relative magnitudes of the flexural and chain stresses are the same. Comparing under these conditions the expressions containing the operators A_0 and A_2, we first of all note that the orders of A_0 and $A_2 h^{-2}$ are the same. Therefore, in comparing terms containing A_2 and A_0, the factor h^4 must be attributed to the terms containing A_2. Then, we note that the order of the ratio $c_0^4 : \mu^1$ equals $a^4 (2k_{max})^4$. Finally, we must bear in mind the numerical coefficients of the summands entering into the sum S_1.

We mentioned above that the terms containing the factor $c_{(2)}$ had to be excluded from the equations by various methods, for example, by a rational choice of the basic surface in the layered shell and in the approximately equivalent homogeneous shell. Here, however, we shall not change the position of the basic surface, but shall directly equate to zero the coefficient of $c_{(2)}$ in eq.(8.31). Then, retaining the dominant terms in S_1 (under the above assumptions), we find the following simplified expression for the sum S_1:

$$S_1 \cong 16h^4(\Delta\omega)^2[A_0(\rho) - 2h^*\rho^*]^2 + 16h^{-4}a^4\left[A_2(2\mu) - \frac{1}{3}(2h^*)^3 2\mu^*\right]^2 + \frac{64}{9}h^{-4}a^4\left[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \frac{1}{3}(2h^*)^3\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]^2. \tag{8.33}$$

As will be seen from the simplified expression for S_1, the terms depending on the chain stresses have been dropped from it. This is a consequence of the assumptions we made on the region of approximation and the neglect of a number of terms entering into eq.(8.31). It is clear that the results obtained from eq.(8.33) are only roughly approximate.

For S_1 to vanish it is sufficient to equate to zero the expressions in the

brackets. Also bearing in mind the condition for the vanishing of the coefficient of $c_{(2)}$ in the right-hand side of eq.(8.31), we find the following /242 simultaneous system of equations:

$$2h^* \rho^* = A_0(\rho), \tag{8.34a}$$

$$\frac{1}{3}(2h^*)^3 \rho^* = A_2(\rho), \tag{8.34b}$$

$$\frac{1}{3}(2h^*)^3 2\mu^* = A_2(2\mu), \tag{8.34c}$$

$$\frac{1}{3}(2h^*)^3 \frac{2\mu^* \lambda^*}{\lambda^* + 2\mu^*} = A_2\left(\frac{2\mu\lambda}{\lambda + 2\mu}\right) \tag{8.34d}$$

Hence, we find

$$2h^* = \sqrt{\frac{3A_2(\rho)}{A_0(\rho)}}; \qquad \rho^* = \frac{A_0^{3/2}(\rho)}{\sqrt{3A_2(\rho)}}; \tag{8.35a}$$

$$2\mu^* = \frac{3A_2(2\mu)\, A_0^{3/2}(\rho)}{[3A_2(\rho)]^{3/2}}; \tag{8.35b}$$

$$\lambda^* = \frac{3A_0^{3/2}(\rho)\, A_2(2\mu)\, A_2\left(\frac{2\mu\lambda}{\lambda + 2\mu}\right)}{\left[A_2(2\mu) - A_2\left(\frac{2\mu\lambda}{\lambda + 2\mu}\right)\right][3A_2(\rho)]^{3/2}}. \tag{8.35c}$$

Of course, if μ^* and λ^* are determined by eqs.(8.35b)-(8.35c), we may in special cases obtain "physically impossible" values of the Poisson constant ν, which may above all indicate the unsuitability of the simplified expression (8.33) for the sum S_1.

Here, we will not investigate the question as to the impossibility of determining the parameters of the homogeneous shell from eq.(8.31).

We shall likewise not investigate in detail the question whether it is permissible to formally apply the equations with "physically impossible" values of Poisson's constants and the physical meaning of such equations. We recall merely that negative values of Poisson's constant correspond to the loss of stability of the shell considered in Sect.6. The "physically impossible" values of Poisson's constant for actual materials may prove to be possible for a /243 medium approximately equivalent to the real medium, and may reflect the specific peculiarities of those problems of mechanics for which this medium has been constructed.

f) Let us consider an example of application of eqs.(8.35a)-(8.35c)*. Let a bimetal shell of thickness 2h consist of a layer of aluminum of 2/3h, adjacent

*This example is merely of an illustrative value.

Table 6

MATERIAL	E, bar	ν	λ, bar	2μ, bar	$\frac{2\mu\lambda}{\lambda+2\mu}$	$\rho\left[\frac{Kq}{cm^3}\right]$	$\Delta\rho$	$\Delta 2\mu$	$\Delta\frac{2\mu\lambda}{\lambda+2\mu}$
ALUMINUM AMG	$0{,}68\cdot10^6$	0,3	$0{,}390\cdot10^6$	$0{,}521\cdot10^6$	$0{,}223\cdot10^6$	$0{,}267\cdot10^{-2}$	$0{,}12\cdot10^{-3}$	$0{,}014\cdot10^6$	$0{,}007\cdot10^6$
DURALUMIN D17	$0{,}70\cdot10^6$	0,3	$0{,}402\cdot10^6$	$0{,}535\cdot10^6$	$0{,}229\cdot10^6$	$0{,}279\cdot10^{-2}$			

to the basic surface, and a layer of duralumin $\frac{4}{3}h$ in thickness. Required, to find $2h^*$, ρ^*, λ^* and μ^*. Table 6 gives the principal physical characteristics of aluminum and duralumin.

From eq.(8.15) we have:

$$A_0(\rho)=\left(0{,}267\cdot10^{-2}+\frac{4}{3}0{,}12\cdot10^{-3}\right)(2h)=0{,}283\cdot10^{-2}(2h),$$

$$A_2(\rho)=\left(\frac{1}{3}\cdot0{,}267\cdot10^{-2}+\frac{26}{81}\cdot0{,}12\cdot10^{-3}\right)(2h)^3=0{,}929\cdot10^{-3}(2h)^3,$$

$$A_2(2\mu)=\left(\frac{1}{3}\cdot0{,}521\cdot10^6+\frac{26}{81}0{,}014\cdot10^6\right)(2h)^3=0{,}178\cdot10^6(2h)^3,$$

$$A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)=\left(\frac{1}{3}\cdot0{,}223\cdot10^6+\frac{26}{81}\cdot0{,}007\cdot10^6\right)(2h)^3=$$
$$=0{,}763\cdot10^5(2h)^3.$$

Making use of eqs.(8.35a)-(8.35c), we find

$$2h^*=0{,}985\,(2h)\ [cm];\qquad \rho^*=0{,}284\cdot10^{-2}\left[\frac{Kq}{cm^3}\right];$$
$$2\mu^*=0{,}544\cdot10^6\ bar;\qquad \lambda^*=0{,}408\cdot10^6\ bar$$

These results show in particular that the values of the elastic constants λ^* and μ^* found here will depend on the density of the layer material.

Comparing the obtained values of $2\mu^*$ and λ^* /244
with those for 2μ and λ given in Table 6, we see that they cannot be termed "averaged", since they are outside the variational limits of 2μ and λ in the material layers. Obviously, this is primarily a consequence of the method of determination adopted here for the reduced thickness of the homogeneous shell, $2h^*$. This thickness was found to be somewhat less than the thickness $2h$ of the layered shell. Besides, the exclusion of the terms depending on the chain stresses also had a considerable influence.

Let us consider a simpler static problem. Of the system of equations (8.34a)-(8.34d) there remain eqs.(8.34c)-(8.34d). Putting $2h^* = 2h$, we find

$$2\mu^* = \frac{3A_2(2\mu)}{(2h)^3}; \qquad \lambda^* = \frac{3A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)A_2(2\mu)}{(2h)^3\left[A_2(2\mu) - A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)\right]}. \tag{8.36}$$

From eqs.(8.36) and the data in Table 6, we find

$$2\mu^* \cong 0{,}535 \cdot 10^6 \ bar; \qquad \lambda^* \cong 0{,}392 \cdot 10^6 \ bar.$$

where the values of $2\mu^*$ and λ^* do not go beyond the limits of the variational interval of 2μ and λ for layer materials almost coinciding with the values of these quantities for duralumin, owing to the fact that the duralumin layer occupies 2/3 of the thickness of the shell and that there are no terms depending on A_0 in the approximate expression (8.33).

g) We remarked repeatedly that it is possible to simplify the sum S_1 by a rational choice of the basic surface in the layered shell and in the equivalent shell. We may, for example, select the basic surface in special cases such that the components $\epsilon_{ik}^{(0)}$ (i, k = 1, 2) on it shall vanish*. This choice of the basic surface, however, is mandatory in solving static problems, since in dynamic problems one must first eliminate the terms with the coefficients $c_{(\bullet)}$, whose meaning is not sufficiently definite in the general case.

h) It is easy to establish that the approximate expression (8.33) for the sum S_1 is also suitable if the quantities ϵ_{ik} (i, k = 1, 2) vanish, since this expression does not contain terms depending on $\epsilon_{ik}^{(0)}$. Of course, if the components $\epsilon_{ik}^{(0)}$ vanish, the accuracy of eq.(8.33) and of the resultant con- /245
sequences increases.

i) In the general case, the quantities λ^*, μ^*, ρ^*, $2h^*$ are determined from the conditions that the right-hand side of eq.(8.31) shall be minimum, i.e., from the conditions:

$$\frac{\partial S_1}{\partial m^*} = 0; \quad \frac{\partial S_1}{\partial G^*} = 0; \quad \frac{\partial S_1}{\partial H^*} = 0; \quad \frac{\partial S_1}{\partial h^*} = 0, \tag{8.37}$$

where

$$m^* = 2h^*\rho^*, \quad G^* = 2h^*\mu^*, \quad H^* = 2h^*\lambda^*.$$

The conditions (8.37) must be associated with the condition (8.32) that

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* Suggestions as to the rational choice of the basic surface are given elsewhere (Bibl.15a, b, 21, 25).

the coefficient of $c_{(2)}$ shall vanish in eq.(8.31).

The system of algebraic equations (8.32) and (8.37) is nonlinear in the required quantities m^*, G^*, H^*, h^*, and h_1. Its solution clearly involves considerable difficulties. We may, for example, use the method of successive approximation, substituting in the second-power terms of eqs.(8.32) and (8.37) the solutions (8.35a) - (8.35c) of the simplified system and determining the next approximation, but in this case there can be no guarantee that the process will converge. Such calculations would be outside the scope of this book.

4. Application of Boundary-Problem Solutions of the Dynamics of Homogeneous Shells to the Construction of a Homogeneous Shell Approximately Equivalent to a Layered Shell

As can be seen from the above, the principal difficulty in solving the problem of the approximation of the Lagrange function L of a layered shell by a Lagrange function L^* of a homogeneous shell lies in the indeterminacy of the variational limits of the variables of the field. This indeterminateness forces us in many cases to consider the approximation over "natural" intervals, determined by the requirement of the applicability of the laws of elasticity theory. Clearly, the introduction of the natural intervals of approximation reduces the accuracy of the results.

The continuous field variables introduced by us yield still another method of constructing a homogeneous shell approximately equal to a layered shell which is free from the above handicap.

Let us assume that we have solved a certain dynamic boundary problem of the theory of homogeneous shells, approximately corresponding in boundary conditions and in loading conditions to the problem of vibrations of a layered shell. We shall assume that in the initial approximation the variable fields are determined from the solution of the problem for the homogeneous shell.

Applying the formula of transformation of the components of tensor quantities, we find according to eqs.(8.17b) - (8.19) the approximate expression for the Lagrange function L for the layered shell with variable coefficients for the operators $A_{\alpha}(f)$ which are functions of the coordinates of the basic surface. /246

The problem of determining the parameters λ^*, μ^*, ρ^*, h^*, and the coordinates of the new basic surfaces in the layered and homogeneous shells reduces to a consideration of the minimum standard deviation

$$I = \int_{\omega_1}^{\omega_2} \int_{0}^{\frac{2\pi}{\omega}} \int_{(S)} (L - L^*)^2 \, dS \, dt \, d\omega, \tag{8.38}$$

where the internal integral extends over the area S of the basic surfaces, common to the layered shell and its equivalent homogeneous shell. The meaning of the remaining notation has been indicated above. The question again reduces

to the solution of a system of nonlinear algebraic equations:

$$\frac{\partial I}{\partial m^*}=0; \quad \frac{\partial I}{\partial G^*}=0; \quad \frac{\partial I}{\partial H^*}=0; \quad \frac{\partial I}{\partial h^*}=0. \tag{8.39}$$

Selecting the new basic surfaces in the layered shell and its approximately equivalent homogeneous shell, we can exclude from the integral I the two dominating terms, and then set up eqs.(8.39).

Let us consider two elementary problems on the equilibrium of a circular closed cylindrical shell of radius R, to illustrate the latter method.

As the first example, let us consider the subcritical axisymmetric deformation of this shell due to the longitudinal compressive forces T uniformly distributed over the lines of intersection of the middle and face surfaces. We shall assume that the boundary conditions do not prevent radial dilatation of the tube.

Confining ourselves to the approximation formulas of the classical theory of shells and to the notations given in Sect.6, we find (Bibl.23d) that the deformed state of the shell is determined here by only a single function $\frac{du}{dx}$. From the condition that the annular forces shall vanish, we obtain

$$w=\nu R\frac{\partial u}{\partial x}. \tag{8.40}$$

We recall that w and $\frac{du}{dx}$ belong to the field variables. We therefore assume that these quantities are the same in the layered and homogeneous shells.

For metals such as steel or aluminum, the Poisson constant differs little from 0.3. Let us put ν = 0.3 in eq.(8.40). /247

In the second elementary problem, let us assume that the longitudinal forces are zero. Then, with the other boundary conditions arbitrary, we find that the deformed state of the shell is described by the function w. Here,

$$\frac{\partial u}{\partial x}=\frac{\nu}{R}w. \tag{8.41}$$

Since eq.(8.19) is set up in a local Cartesian system of coordinates, let us consider the formulas for the direct and inverse transitions between the internal coordinates of the shell and the local Cartesian system. These formulas are of the following form:

$$x_1=x-x_0;\ x_2=(R-z)\sin\frac{s-s_0}{R};\ x_3=R-(R-z)\cos\frac{s-s_0}{R}; \tag{8.42a}$$

$$x^1 = x = x_1 + x_0;\ x^2 = s = s_0 + R\tan^{-1}\frac{x_2}{R - x_3};$$

$$x^3 = z = R - \sqrt{(x_3 - R)^2 + x_2^2}. \tag{8.42b}$$

where x_i are the local Cartesian coordinates and x^i the internal coordinates of the shell. In this case, the internal coordinates are determined by eqs.(6.1). The coordinates x_0 and s_0 determine the position of the origin of the coordinate bases of the local Cartesian system on the middle surface of the shell.

Making use of eqs.(I, 5.17), let us express the displacement vector components in the local Cartesian system of coordinates in terms of the components u and w. We obtain

$$u_1 = u\frac{\partial x}{\partial x_1};\quad u_2 = w\frac{\partial z}{\partial x_2};\quad u_3 = w\frac{\partial z}{\partial x_3}. \tag{8.42c}$$

On the basis of eqs.(8.42b) - (8.42c) and putting $x = x_0$ and $s = s_0$ after differentiation, we find all the quantities characterizing the axisymmetric stress-strain state of the shell entering into eq.(8.19).

Since eqs.(8.40) - (8.41) were obtained from the classical theory of shells, we must make use of them in determining the quantities $\epsilon_{ik}^{(m)}$. We obtain

$$\varepsilon_{11}^{(0)} = \frac{\partial u}{\partial x};\quad \varepsilon_{11}^{(1)} = -\frac{d^2 w}{dx^2};\quad \varepsilon_{11}^{(2)} = 0;\quad \varepsilon_{22}^{(0)} = -\frac{w}{R};$$

$$\varepsilon_{22}^{(1)} = -\frac{w}{R^2};\quad \varepsilon_{22}^{(2)} = -\frac{w}{R^3};\quad \varepsilon_{12}^{(0)} = \varepsilon_{12}^{(1)} = \varepsilon_{12}^{(2)} = 0. \tag{8.43a}$$

Further, on the basis of the Kirchhoff-Love hypotheses, we put /248

$$\tau_{i3}^{(m)} = 0$$
$$(i = 1,\ 2,\ 3;\ m = 0,\ 1,\ 2). \tag{8.43b}$$

Of course, it is also possible to use equations that do not rely on the Kirchhoff-Love hypotheses, but the illustrative character of the examples considered here and the general object of constructing an approximate solution do not justify the complications which these methods would involve.

Let us consider the first elementary problem. In axisymmetric compression of a cylindrical shell under the conditions of free radial dilatation, the

flexural moments are zero, so that the derivative $\frac{d^2w}{dx^2}$ also vanishes. From eqs.(8.43a) and (8.40) it follows that

$$\varepsilon_{11}^{(0)} = \frac{du}{dx}; \quad \varepsilon_{11}^{(1)} = \varepsilon_{11}^{(2)} = 0; \quad \varepsilon_{22}^{(0)} = -\nu \frac{du}{dx}; \quad \varepsilon_{22}^{(1)} = -\frac{\nu}{R}\frac{au}{dx};$$

$$\varepsilon_{22}^{(2)} = -\frac{\nu}{R^2}\frac{du}{dx}; \quad \varepsilon_{12}^{(0)} = \varepsilon_{12}^{(1)} = \varepsilon_{12}^{(2)} = 0; \tag{8.44a}$$

$$\theta^{(0)} = (1-\nu)\frac{du}{dx}; \quad \theta^{(1)} = -\frac{\nu}{R}\frac{du}{dx}; \quad \theta^{(2)} = -\frac{\nu}{R^2}\frac{du}{dx}. \tag{8.44b}$$

Making use of eq.(8.19) and noting that, in static problems, the Lagrange function L equals $-\Pi$, we find

$$4I = \left\{\left[A_0\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - (2h)\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right](1-\nu)^2 - 2\left[A_1\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \right.\right.$$
$$\left. - \frac{1}{2}(2h)^2\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]\frac{\nu(1-\nu)}{R} + \left[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \frac{1}{3}(2h)^3 \times \right.$$
$$\left. \times \frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]\frac{2\nu^2-\nu}{R^2} + [A_0(2\mu) - (2h)2\mu^*](1+\nu^2) +$$
$$\left. + 2\left[A_1(2\mu) - \frac{1}{2}(2h)^2 2\mu^*\right]\frac{\nu^2}{R} + 2\left[A_2(2\mu) - \frac{1}{3}(2h)^3 2\mu^*\right]\frac{\nu^2}{R^2}\right\}^2 \times$$
$$\times \int_{(S)} \left(\frac{du}{dx}\right)^4 dS. \tag{8.45}$$

Here, it has been assumed that $h^* = h$. If we also assume that $\nu \cong 0.3$, then $\lambda^* \cong \frac{3}{2}\mu^*$. This permits us to find μ^*, by equating to zero the expression in the braces in eq.(8.45)*. The equation determining μ^* will be linear. The result will not depend on the function u. Rejecting the terms that depend on the ratio 2h : R and approximately setting the factor $(1 - \nu)^2$ as equal to 0.5, we find /249

$$\mu^* \cong \frac{35}{92}\frac{1{,}1A_0(2\mu) + 0{,}5A_0\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)}{2h} \tag{8.46}$$

* In this case I reaches its exact lower boundary.

The methods of further refinement in this case are so obvious that we shall not discuss them here. Equation (8.46) determines μ^* by means of an operation close to a simple averaging of the elastic constant of the layers over the thickness of the shell.

Consider the second elementary case. Making use of eqs.(III, 10.1a) - (III, 10.3a), and (8.43a), we find

$$\varepsilon_{11}^{(0)}=\frac{\nu}{R}w;\quad \varepsilon_{11}^{(1)}=-\frac{d^2w}{dx^2};\quad \varepsilon_{11}^{(2)}=0;\quad \varepsilon_{22}^{(0)}=-\frac{w}{R};$$

$$\varepsilon_{22}^{(1)}=-\frac{w}{R^2};\quad \varepsilon_{22}^{(2)}=-\frac{w}{R^3};\quad \varepsilon_{12}^{(0)}=\varepsilon_{12}^{(1)}=\varepsilon_{12}^{(2)}=0; \tag{8.47a}$$

$$\theta^{(0)}=-(1-\nu)\frac{w}{R};\quad \theta^{(1)}=-\frac{d^2w}{dx^2}-\frac{w}{R^2};\quad \theta^{(2)}=-\frac{w}{R^3}. \tag{8.47b}$$

Here we confined ourselves to the approximations corresponding to the Kirchhoff-Love hypotheses. From eqs.(8.19) and (8.38) we find

$$\begin{aligned}
I=\frac{1}{4}\int\limits_{(S)}\Bigg\{\Bigg\{\Bigg[&A_0\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)-(2h)\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\Bigg](1-\nu)^2+\\
&+2\Bigg[A_1\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)-\frac{1}{2}(2h)^2\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\Bigg]\frac{(1-\nu)}{R}+\\
&+\Bigg[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)-\frac{1}{3}(2h)^3\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\Bigg]\frac{(2-\nu)}{R^2}+[A_0(2\mu)-\\
&-(2h)2\mu^*](1+\nu^2)+\frac{2}{R}\Bigg[A_1(2\mu)-\frac{1}{2}(2h)^2 2\mu^*\Bigg]+\frac{2}{R^2}\Bigg[A_2(2\mu)-\\
&-\frac{1}{3}(2h)^3 2\mu^*\Bigg]\Bigg\}\frac{w^2}{R^2}+2\Bigg\{\Bigg[A_1\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)-\frac{1}{2}(2h)^2\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\Bigg]\times\\
&\times(1-\nu)+\frac{1}{R}\Bigg[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)-\frac{1}{3}(2h)^3\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\Bigg]-\Bigg[A_1(2\mu)-\\
&-\frac{1}{2}(2h)^2 2\mu^*\Bigg]\nu\Bigg\}\frac{w}{R}\frac{d^2w}{dx^2}+\Bigg\{\Bigg[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right)-\frac{1}{3}(2h)^3\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\Bigg]+\\
&+\Bigg[A_2(2\mu)-\frac{1}{3}(2h)^3 2\mu^*\Bigg]\Bigg\}\left(\frac{d^2w}{dx^2}\right)^2\Bigg\}dS.
\end{aligned} \tag{8.48}$$

Here, as above, we have assumed that $h^* = h$.

If we put $\nu^* \cong 0.3$ and, consequently, $\lambda^* \cong \frac{3}{2}\mu^*$, and also neglect the /250
terms containing the ratio $w : R$, then we find from eq.(8.48), on equating I to zero,

$$\mu^* = \frac{21}{20}\,\frac{A_2(2\mu) + A_2\left(\dfrac{2\mu\lambda}{\lambda + 2\mu}\right)}{(2h)^3}. \tag{8.49}$$

We omit a comparison of the numerical values of μ^* determined from eqs.(8.48) and (8.49).

The determination of μ^* on the basis of eq.(8.48) can be further refined by assuming, for example, that w is expressed by

$$w = A_m \sin\frac{m\pi x}{l} \tag{8.50a}$$

and, consequently, that

$$\frac{d^2 w}{dx^2} = -\left(\frac{m\pi}{l}\right)^2 w. \tag{8.50b}$$

In this case,

$$\begin{aligned}
I = \frac{1}{4}\Bigg\{\Bigg\{&\left[A_0\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - (2h)\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right](1-\nu)^2 + \\
&+2\left[A_1\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \frac{1}{2}(2h)^2\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]\frac{(1-\nu)}{R} + \left[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \right.\\
&\left. - \frac{1}{3}(2h)^3\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right]\frac{(2-\nu)}{R^2} + [A_0(2\mu) - (2h)\,2\mu^*](1+\nu^2) + \\
&+\frac{2}{R}\left[A_1(2\mu) - \frac{1}{2}(2h)^2\,2\mu^*\right] + \frac{2}{R^2}\left[A_2(2\mu) - \frac{1}{3}(2h)^3 2\mu^*\right]\Bigg\}\frac{1}{R^2} - \\
&-2\left\{\left[A_1\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \frac{1}{2}(2h)^2\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right](1-\nu) + \right.\\
&+\frac{1}{R}\left[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \frac{1}{3}(2h)^3\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right] - \left[A_1(2\mu) - \right.\\
&\left.\left. - \frac{1}{2}(2h)^2 2\mu^*\right]\nu\right\}\frac{1}{R}\left(\frac{m\pi}{l}\right)^2 + \left\{\left[A_2\left(\frac{2\mu\lambda}{\lambda+2\mu}\right) - \frac{1}{3}(2h)^3 \times \right.\right.\\
&\left.\left.\times\frac{2\mu^*\lambda^*}{\lambda^*+2\mu^*}\right] + \left[A_2(2\mu) - \frac{1}{3}(2h)^3\,2\mu^*\right]\right\}\left(\frac{m\pi}{l}\right)^4\Bigg\}\iint_{(S)} w^4\,dS.
\end{aligned} \tag{8.51}$$

Equating I to zero and taking $\nu^* = 0.3$, we find μ^* as a function of the ratio $m : \ell$. At high values of m we again arrive at eq.(8.49).

Section 9. Construction of the Approximate Solution to Problems of the Dynamics of Layered Shells. Application of the Method of Perturbations and Nonremovable Errors /251

An analysis of the problem of the approximate replacement of a layered shell by a homogeneous shell shows that this leads to considerable difficulties. The major difficulty is the absence of means permitting us to establish the universal region of approximation of the Lagrange function L of the layered shell by the Lagrange function L* of the homogeneous shell, which frequently forces us to turn to the "natural" region of approximation determined by the requirement that all quantities shall vary within the region of applicability of the linear theory of elasticity.

The absence of individualization of the variational intervals of the principal variables may decrease the accuracy of the results, particularly when special problems are being considered. Even the absence of individualization of the variational regions of the variables must be considered as a preliminary hypothesis on the stressed state of the shell, which we have emphasized in our derivation of eqs.(8.31), (8.33) and of formulas (8.35a) - (8.36). For this reason, the method indicated in Sect.8.4 is the most justified. This method is based on a preliminary consideration of the solutions of concrete boundary problems, although in a number of cases this method may lead to unwieldy calculations, specifically in the solution of systems of nonlinear algebraic equations, as noted above.

Let us consider now the general order of the approximate solution of problems of the dynamics of layered shells.

The first stage of the solution consists in the construction of a homogeneous shell approximately equivalent to the layered shell. Here it is most expedient to start from the solution of the boundary problem for the homogeneous shell. This boundary problem should be so selected that, for the prescribed loads and boundary conditions, it shall exactly or approximately correspond to the problem of the dynamics of the layered shell. Then we must use the technique given in Sect.8.4 for determining the quantities characterizing the homogeneous shell.

If this method involves unnecessary difficulties, we must use the procedure indicated in Sect.8.3. In determining the variational intervals of the field variables, we must also attempt to match these and all other assumptions, necessary for deriving a weighted sum of the form of eq.(8.31), with the conditions of the concrete problem of mechanics. Obviously, the first stage of solution of the problem has practical meaning only if the difference between the values of the physical constants of the layer materials is substantial. If these differences are slight, we may confine ourselves to the weighted averaging of the physical constants over the thickness of the shell, taking the weights equal to the thicknesses of the corresponding shells. However, our analysis shows that such a method of determining the physical constants, /252
while quite logical at first glance, does not correspond to the optimum quad-

ratic approximation of the function L by the function L*.

The first stage of the approximate solution is completed by determining, from the solution of the boundary problem, the field variables in the homogeneous shell. These variables, as noted at the beginning of Sect.8, approximately represent the field variables of the layered shell. Then, using these field variables, we construct the fields of displacements, strains, and stresses in the layered shell according to eqs.(8.3) - (8.10b). In this case, we must first establish the relation between the coordinate z* in the homogeneous shell and the coordinate z in the layered shell. This relation may be taken in the following form:

$$\frac{z}{z^*} = \frac{h}{h^*}. \tag{9.1}$$

Equation (9.1) is not connected with the equations of motion nor with the boundary conditions. This equation was taken arbitrarily by us, as the simplest form of the relation between z and z*. We may evidently make use of this arbitrariness to improve the approximate solutions sought. We will not further investigate this question here.

The construction of the fields of the principal tensor quantities completes the second stage of the approximate solution of the problem of the dynamics of layered shells.

We recall that the solution under consideration is based on an approximation involving a finite segment of the frequency spectrum. By enlarging this segment, we increase the "weight" of the terms depending on the kinetic energy, as is shown for example, by eq.(8.31), and thus worsen the approximation of the quantities depending on the potential energy. These quantities are the components of the strain and stress tensors. Thus, the approximate solution constructed by us, as was to be expected, will have only limited value.

Let us refine the meaning of the approximate solution considered here, by comparing it with the exact solution of the linear theory of elasticity.

Assume, for definiteness, that the problem for the approximately equivalent homogeneous shell has been solved by the first method of reduction considered in Chapter III, i.e., by the method of expansion in tensor series in powers of z. Then, in the homogeneous shell, the conditions on the boundary surfaces will be satisfied as well as the relations expressing Hooke's law and Saint-Venant's compatibility conditions. The equations of motion and the equations of the contour surfaces will be approximately satisfied. The relative accuracy of satisfaction of the boundary conditions on the contour surfaces will be lower than the relative accuracy of representation of the components of the displacement vector and of the strain tensor by the segments of the /253
tensor series.

The fields of displacements and stresses in the layered shells, constructed in the second stage, satisfy the conditions on the boundary surfaces, the conditions on the interfaces between the layers, the equations resulting from

the generalized Hooke's law, and Saint-Venant's compatibility conditions. The equations of motion and the boundary conditions on the contour surfaces are approximately satisfied. The error in the satisfaction of the equations of motion and of the conditions on the contour surfaces will depend, in this case, not only on the rejected terms of the series representing the displacement vector components, but also on the differences between the physical constants of the layer materials and on the difference between the thickness of the layered shell $2h$ and the thickness of the equivalent shell $2h^*$.

The error can be decreased by applying the method of perturbations. In fact, knowing approximately the stresses on the boundaries of the layers, we can now consider separately the motion of each layer. Each of the layers performs a motion under the action of loads on the surface and of quasi-body forces, which can be found by substituting the approximately determined components of the displacement vector into the equations of motion (II, 5.5a or 5.5b).

We can then apply one of the systems of equations of motion of homogeneous shells, considered in Chapter III, to each layer separately. This will permit to eliminate part of the error arising as a result of the differences between the quantities h, ρ, λ, μ in the layers of the shell and the quantities h^*, ρ^*, λ^*, μ^* in the homogeneous shell. There still remains, however, the error depending on the approximate determination of the surface forces on the boundaries of the layers of this shell. This error is connected with the fact that the function L^* only approximately represents the function L. For this reason, the "exact" subdivision of the general problem of motion of a layered shell into isolated problems of motion of its layers cannot be carried out, and the latter error will be irremovable.

Further studies, going beyond the scope of this book, must obviously center on the search for means of decreasing this irremovable error.

Section 10. Application of Optimum Quadratic Approximations to the Problem of Reduction of the Three-Dimensional Problem of the Elasticity Theory to the Two-Dimensional Problem

The methods used by us in solving the problem of constructing a system which, according to some criterion, is approximately equivalent to the prescribed system are also applicable to the problems considered in Chapter III. We have stated above that the reduction of the three-dimensional problem of the theory of elasticity to a two-dimensional problem of the theory of shells can be regarded as the construction of a system approximately equivalent to /254
a three-dimensional elastic body. The objects of the above-selected approximation were the potential strain energy and the Lagrange function of the elastic body, using systems of variables that yield the explicit analytic expressions of the approximated and approximating functions.

The variables used previously are unsuitable for solution of the reduction problem. This forces us to abandon the potential strain energy and the Lagrange function of an elastic body as objects of approximation. As the objects of the approximations, let us select prescribed body and surface forces and prescribed displacements on certain portions of the surface of the body.

We shall confine ourselves to a discussion of the general principles of the proposed method and not go into the details. We will start from the nonlinear Lamé equations in the form of (II, 7.6), assuming for simplicity that the system of coordinate axes of the undeformed basic surface of the shell coincides with its lines of curvature.

Let us select two coordinate lines as the lines of origin of the coordinate net. We shall define the position of an arbitrary point M on the basic surface by its arc coordinates s^i, equal to the absolute values of the distances of the point M from the lines of origin. These distances are measured along the coordinate lines passing through the point M from point M to points M_i of their intersection with the origin lines of the coordinate net*. The s^i coordinates are connected with the x^j coordinates by means of curvilinear integrals taken along the coordinate axes.

$$s^i = \int\limits_{x_0^i}^{x^i} \sqrt{g_{ii}(x^1, x^2)}\, dx^i \qquad (i = 1,\ 2) \tag{10.1}$$

(do not sum over i!). With a coordinate system selected in this manner, the metric tensor on the basic surface will have the following components:

$$g_{ik} = g^{ik} = \delta^i_k, \tag{10.2}$$

where δ^i_k is the Kronecker delta.

It follows from eq.(II, 7.6) that, in the system of coordinates selected by us, the covariant components of the body forces may be represented in the following form:

$$\rho F_i = \rho \frac{\partial^2 u_i}{\partial t^2} - \mu \sum_{k=1}^{3} \partial_k^2 u_i - (\lambda + \mu) \sum_{k=1}^{3} \partial_i \partial_k u_k +$$

$$+ M^{pr}_{..i} \partial_p u_r + N^k_{.i} u_k - \Phi_i \qquad (i,\ k,\ p,\ r = 1,\ 2,\ 3). \tag{10.3}$$

In eqs.(10.3), $M^{pr}_{..i}(s^j, z)$ and $M^k_{..i}(s^j, z)$ are functions of the coordinates s^j of the undeformed basic surface and of the coordinate z, depending on the Christoffel symbols Γ^i_{jk} and on their first derivatives with respect to s^i and z (j = 1, 2). /255

The functions Φ_i are components of the additional body forces considered in (II, Sect.7). These functions, as results from (II, 7.4), have the follow-

* For details, see (Bibl.6, pp.101-102).

ing composition:

$$\Phi_i = \Phi_i(s^j,\ z,\ \partial_p\partial_q u_r,\ \partial_q u_k,\ u_j). \tag{10.4}$$

Here, all indices except j take the values 1, 2, 3. We shall not write out the expressions for these functions in expanded form, since we intend hereafter to confine ourselves only to a discussion of the general principles of constructing the system of equations of the theory of shells by means of the method considered here. In exactly the same manner, we can derive the components of the surface forces on the boundary surfaces of the shell and on its contour surfaces.

Before considering the conditions on the boundary surfaces, let us assume that the surface forces acting on them and the stress tensor components corresponding to them have undergone parallel displacement to the basic surface in the sense of Levi-Civita. Making use of (II, 8.13) we find that the following conditions are satisfied on the boundary surfaces of the shell:

$$\begin{gathered}\pm[1+\psi_0(s^j,\ z^{(\pm)},\ \partial_p u_q,\ u_r)]\,\tau_{i3}^{(\pm)} = X_{(\pm)i}\\ (i,\ p,\ q,\ r=1,\ 2,\ 3;\ j=1,\ 2).\end{gathered} \tag{10.5}$$

where the scalar ψ_0 is determined from (II, 8.12); the stress tensor components displaced to the basic surface are denoted as in Chapter III; the sign (+) corresponds to a boundary surface on which the direction of the unit vector $\vec{n}_0$ of the external normal coincides with the direction of the vector $\vec{e}_3$ on the undeformed basic surface, while the sign (-) corresponds to a boundary surface on which these directions are opposite.

In eqs.(10.5) the coordinate z has a fixed value. Let

$$z^{(+)} = h_2;\ z^{(-)} = -h_1. \tag{10.6}$$

These equations determine the position of the boundary surfaces for a prescribed position of the basic surface within the shell. We shall hereafter assume that h_1 and h_2 are constants. On the contour surfaces, the following relations are satisfied:

$$[1+\psi_0(s^j,\ z,\ \partial_p u_q,\ u_r)]\,n_0^j\tau_{ij} = f_i. \tag{10.7}$$

The stress tensor components are connected with the displacement vector components by the equations resulting from Hooke's law (II, 4.3): /256

$$\begin{gathered}\tau_{ij} = \lambda\delta_j^i(\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3) + \mu(\partial_i u_j + \partial_j u_i) + \\ + L^p_{.ij}(s^j,\ z)\,u_p + \Phi_{ij}(s^j,\ z,\ \partial_p u_q,\ u_r),\end{gathered} \tag{10.8a}$$

$$\tau_{i3} = \mu(\partial_i u_3 + \partial_3 u_i) + L^p_{.i3}(s^j, z)\, u_p + \Phi_{i3}(s^j, z, \partial_p u_q, u_r), \tag{10.8b}$$

$$\tau_{33} = \lambda(\partial_1 u_1 + \partial_2 u_2) + (\lambda + 2\mu)\partial_3 u_3 + L^p_{.33}(s^j, z)\, u_p + \Phi_{33}(s^j, z, \partial_p u_q, u_r)$$

$$(i, j = 1, 2;\ p, q, r = 1, 2, 3). \tag{10.8c}$$

where the functions $L^p_{.ij}$, $L^p_{.i3}$, $L^p_{.33}$ are expressed in terms of the Christoffel symbols $\Gamma^p_{.ij}$, $\Gamma^p_{.i3}$, $\Gamma^p_{.33}$. The functions Φ_{ij}, Φ_{i3} and Φ_{33} are nonlinear in the displacement vector components and their derivatives with respect to the coordinates s^j and z of terms entering into Hooke's law (II, 4.3) whose composition may be established, for instance, from eq.(II, 7.2). These functions have a similar meaning in the case of physical nonlinearity, i.e., of a Hooke's law determined by (II, 4.7).

Section 11. Approximate Expressions of the Displacement Vector Components and the Equations of Motion of the Shell

Let us assume that the displacement vector components, displaced to the basic surface, can be represented by approximation formulas similar to those considered in Chapter III.

$$u_j \approx \sum_{m=0}^{N} \varphi_m(z)\, u_j^{(m)}(t, s^j). \tag{11.1}$$

where $\varphi_m(z)$ are prescribed functions $\left(\text{above, we mostly used } \varphi_m(z) = \frac{z^m}{m!}\right)$ while the coefficients $u_j^{(m)}$ are unknown functions to be determined.

The right-hand side of eq.(11.1) contains a finite sum such that, here as above, the number of degrees of freedom of the shell in the direction of the coordinate $x^3 = z$ is restricted. Substituting eqs.(11.1) into eq.(10.3), we find the following approximate expressions for the components of the body forces:

$$\rho F_i^* = \sum_{m=0}^{N} \Big\{ \varphi_m(z) \Big[\rho \frac{\partial^2 u_i^{(m)}}{\partial t^2} - \mu \sum_{k=1}^{2} \partial_k^2 u_i^{(m)} - (\lambda + \mu) \times$$

$$\times \sum_{k=1}^{2} \partial_i \partial_k u_k^{(m)} + M^{kp}_{..i} \partial_k u_p^{(m)} + N^p_{.i} u_p^{(m)} \Big] + \varphi_m'(z) [\dot{M}^{3p}_{..i} u_p^{(m)} -$$

$$- (\lambda + \mu)\partial_i u_3^{(m)}] - \mu \varphi_m''(z)\, u_i^{(m)} \Big\} - \Psi_i(s^j, z, \partial_j \partial_k u_p^{(m)}, \partial_k u_q^{(m)}, u_r^{(m)});$$

(11.2a)

$$\rho F_3^* = \sum_{m=0}^{N} \left\{ \varphi_m(z) \left[\rho \frac{\partial^2 u_3^{(m)}}{\partial t^2} - \mu \sum_{k=1}^{2} \partial_k^2 u_3^{(m)} + M^{kp}_{..3} \partial_k u_p^{(m)} + \right. \right.$$

$$\left. + N^p_{.3} u_p^{(m)} \right] + \varphi'_m(z) \left[M^{3p}_{..3} u_p^{(m)} - (\lambda + \mu) \sum_{k=1}^{2} \partial_k u_k^{(m)} \right] -$$

$$\left. - (\lambda + 2\mu) \varphi''_m(z) u_3^{(m)} \right\} - \Psi_3(s^j, z, \partial_j \partial_k u_p^{(m)}, \partial_k u_q^{(m)}, u_r^{(m)})$$ /257

$$(i, j, k = 1, 2; p, q, r = 1, 2, 3). \qquad (11.2b)$$

The functions Ψ_1 and Ψ_3 are the results of substitution, into the functions Φ_1 and Φ_3, of the approximate eqs.(11.1) for the displacement vector components. If we confine ourselves to studying weakly nonlinear problems, then the functions Ψ_1 and Ψ_3 will be polynomials of the functions φ_m and their derivatives.

Let us now consider the approximate expressions for the surface forces on the boundary surfaces of the shell. Making use of eqs.(10.5) and (10.8a) - (10.8c), we obtain

$$X^*_{(\pm)i} = \pm \sum_{m=0}^{N} \{ \varphi_m(z^{(\pm)}) [\mu \partial_i u_3^{(m)} + L^p_{.i3}(s^j, z^{(\pm)}) u_p^{(m)}] +$$

$$+ \mu \varphi'_m(z^{(\pm)}) u_i^{(m)} \} + \Psi_{i3}(s^j, z^{(\pm)}, \partial_k u_p^{(m)}, u_q^{(m)}); \qquad (11.3a)$$

$$X^*_{(\pm)3} = \pm \sum_{m=0}^{N} \{ \varphi_m(z^{(\pm)}) [\lambda (\partial_1 u_1^{(m)} + \partial_2 u_2^{(m)}) + L^p_{.33}(s^j, z^{(\pm)}) u_p^{(m)}] +$$

$$+ (\lambda + 2\mu) \varphi'_m(z^{(\pm)}) u_3^{(m)} \} + \Psi_{33}(s^j, z^{(\pm)}, \partial_k u_p^{(m)}, u_q^{(m)})$$

$$(i, j, k = 1, 2; p, q, r = 1, 2, 3). \qquad (11.3b)$$

The quantities $z^{(\pm)}$ are determined from eqs.(10.6). The functions

$$\Psi_{i3}(s^j, z^{(\pm)}, \partial_k u_p^{(m)}, u_q^{(m)}) = \Psi^{(\pm)}_{i3} \text{ and } \Psi_{33}(s^j, z^{(\pm)}, \partial_k u_p^{(m)}, u_q^{(m)}) = \Psi^{(\pm)}_{33}$$

contain the set of nonlinear terms entering into the surface forces $X^*_{(\pm)i}$ and $X^*_{(\pm)3}$.

Let us set up the functional:

$$I = \int_0^{t_1} \iint_{(S)} \left\{ \int_{-h_1}^{+h_2} [(\rho F_1 - \rho F_1^*)^2 + (\rho F_2 - \rho F_2^*)^2 + (\rho F_3 - \rho F_3^*)^2](1 - k_1 z) \times \right.$$

$$\times (1 - k_2 z)\, dz + (X_{(+)1} - X^*_{(+)1})^2 + (X_{(-)1} - X^*_{(-)1})^2 +$$

$$+ (X_{(+)2} - X^*_{(+)2})^2 + (X_{(-)2} - X^*_{(-)2})^2 + (X_{(+)3} - X^*_{(+)3})^2 +$$

$$+ (X_{(-)3} - X^*_{(-)3})^2 \Big\}\, dS\, dt. \tag{11.4}$$

where t_1 is an arbitrary instant of time and the integral $\int\limits_{(S)}$ extends over the area of the basic surface. The other symbols are known from Chapter III. /258

We shall determine the generalized coordinates $u_p^{(m)}$ of a system replacing the shell from the conditions of the minimum of the functional I. It is well known that here we may use the direct methods, for example the Ritz method and the classical method, by setting up the Euler-Lagrange-Ostrogradskiy equations*. We shall not consider the Ritz method but discuss the Euler-Lagrange-Ostrogradskiy equations. Let us introduce the notation:

$$2W = \int\limits_{-h_1}^{+h_2} [(\rho F_1 - \rho F_1^*)^2 + (\rho F_2 - \rho F_2^*)^2 + (\rho F_3 - \rho F_3^*)^2]\,(1 - k_1 z) \times$$

$$\times (1 - k_2 z)\, dz + (X_{(+)1} - X^*_{(+)1})^2 + (X_{(-)1} - X^*_{(-)1})^2 +$$

$$+ (X_{(+)2} - X^*_{(+)2})^2 + (X_{(-)2} - X^*_{(-)2})^2 + (X_{(+)3} - X^*_{(+)3})^2 +$$
$$+ (X_{(-)3} - X^*_{(-)3})^2. \tag{11.5}$$

The Euler-Lagrange-Ostrogradskiy equations can be represented in the following form:

$$\frac{\partial W}{\partial u_p^{(m)}} - \sum_{j=1}^{2} \frac{\partial}{\partial s^j} \frac{\partial W}{\partial (\partial_j u_p^{(m)})} + \sum_{j=1}^{2} \sum_{k=1}^{2} \frac{\partial^2}{\partial s^j \partial s^k} \frac{\partial W}{\partial (\partial_j \partial_k u_p^{(m)})} +$$
$$+ \frac{\partial^2}{\partial t^2} \frac{\partial W}{\partial \left(\frac{\partial^2 u_p^{(m)}}{\partial t^2} \right)} = 0$$
$$(p = 1, 2, 3;\ m = 0, 1, 2 \ldots, N). \tag{11.6a}$$

* Cf., for example, V.I.Smirnov, Course in Higher Mathematics, Vol.IV, Gostekhizdat, 1951.

We shall consider the left-hand side of eq.(11.6a) as the functional derivative $\frac{\delta W}{\delta u_p^{(m)}}$. Then the system of eq.(11.6a) can be replaced by the short formula

$$\frac{\delta W}{\delta u_p^{(m)}}=0. \tag{11.6b}$$

Equations (11.6a) or (11.6b) approximately determine the motion of an element of the shell. Making use of eq.(11.5) for the function W, let us put eqs.(11.6a) into the following form: /259

$$\frac{1}{2}\int_{-h_1}^{h_2}\left[\sum_{q=1}^{3}\frac{\delta(\rho F_q-\rho F_q^*)^2}{\delta u_p^{(m)}}\right](1-k_1z)(1-k_2z)\,dz+$$

$$+\frac{1}{2}\sum_{q=1}^{3}\frac{\delta(X_{(+)q}-X_{(+)q}^*)^2}{\delta u_p^{(m)}}+\frac{1}{2}\sum_{q=1}^{3}\frac{\delta(X_{(-)q}-X_{(-)q}^*)^2}{\delta u_p^{(m)}}=0, \tag{11.7a}$$

or

$$\int_{-h_1}^{h_2}\left\{\sum_{q=1}^{3}\left[(\rho F_q-\rho F_q^*)\frac{\partial(\rho F_q^*)}{\partial u_p^{(m)}}-\sum_{j=1}^{2}\frac{\partial}{\partial s^j}\left[(\rho F_q-\rho F_q^*)\frac{\partial(\rho F_q^*)}{\partial(\partial_j u_p^{(m)})}\right]+\right.\right.$$

$$+\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{\partial^2}{\partial s^j\partial s^k}\left[(\rho F_q-\rho F_q^*)\frac{\partial(\rho F_q^*)}{\partial(\partial_j\partial_k u_p^{(m)})}\right]+\frac{\partial^2}{\partial t^2}\left[(\rho F_q-\rho F_q^*)\times\right.$$

$$\left.\left.\left.\times\frac{\partial(\rho F_q^*)}{\partial\left(\frac{\partial^2 u_p^{(m)}}{\partial t^2}\right)}\right]\right]\right\}(1-k_1z)(1-k_2z)\,dz+\sum_{q=1}^{3}\left\{(X_{(\pm)q}-X_{(\pm)q}^*)\times\right.$$

$$\left.\times\frac{\partial X_{(\pm)q}^*}{\partial u_p^{(m)}}-\sum_{j=1}^{2}\frac{\partial}{\partial s^j}\left[(X_{(\pm)q}-X_{(\pm)q}^*)\frac{\partial X_{(\pm)q}^*}{\partial(\partial_j u_p^{(m)})}\right]\right\}=0$$

$$(p=1,\ 2,\ 3;\ m=0,\ 1,\ 2,\ \ldots,\ N). \tag{11.7b}$$

Let us now consider in more detail eqs.(11.7b), separating their linear

part. We shall not write out the nonlinear terms in their expanded form, since they are too unwieldy. Making use of eqs.(11.2a) - (11.3b), we find

$$\frac{\partial(\rho F_i^*)}{\partial u_p^{(m)}}=\varphi_m(z)\,N^p_{.i}+\varphi_m'(z)\,M^{3p}_{..i}-\mu\varphi_m''(z)\,\delta_i^p-\frac{\partial\Psi_i}{\partial u_p^{(m)}}; \tag{11.8a}$$

$$\frac{\partial(\rho F_3^*)}{\partial u_p^{(m)}}=\varphi_m(z)\,N^p_{.3}+\varphi_m'(z)M^{3p}_{..3}-(\lambda+2\mu)\varphi_m''(z)\,\delta_3^p-\frac{\partial\Psi_3}{\partial u_p^{(m)}}; \tag{11.8b}$$

$$\frac{\partial(\rho F_i^*)}{\partial(\partial_j u_p^{(m)})}=\varphi_m(z)\,M^{kp}_{..i}\delta_k^j-(\lambda+\mu)\,\varphi_m(z)\,\delta_j^i\delta_3^p-\frac{\partial\Psi_i}{\partial(\partial_j u_p^{(m)})}; \tag{11.8c}$$

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$$\frac{\partial(\rho F_3^*)}{\partial(\partial_j u_p^{(m)})}=\varphi_m(z)\,M^{kp}_{..3}\,\delta_k^p-(\lambda+\mu)\,\varphi_m(z)\,\delta_j^k\,\delta_k^p-\frac{\partial\Psi_3}{\partial(\partial_j u_p^{(m)})}; \tag{11.8d}$$

$$\frac{\partial(\rho F_i^*)}{\partial(\partial_j\partial_k u_p^{(m)})}=-\mu\varphi_m(z)\,\delta_i^p\,\delta_j^k-(\lambda+\mu)\,\varphi_m(z)\,\delta_j^i\,\delta_k^p-\frac{\partial\Psi_i}{\partial(\partial_j\partial_k u_p^{(m)})}; \tag{11.8e}$$

$$\frac{\partial(\rho F_3^*)}{\partial(\partial_j\partial_k u_p^{(m)})}=-\mu\varphi_m(z)\,\delta_3^p\,\delta_j^k-\frac{\partial\Psi_3}{\partial(\partial_j\partial_k u_p^{(m)})}; \tag{11.8f}$$

$$\frac{\partial(\rho F_i^*)}{\partial\left(\dfrac{\partial^2 u_p^{(m)}}{\partial t^2}\right)}=\rho\varphi_m(z)\,\delta_i^p;\qquad \frac{\partial(\rho F_3^*)}{\partial\left(\dfrac{\partial^2 u_p^{(m)}}{\partial t^2}\right)}=\rho\varphi_m(z)\,\delta_3^p; \tag{11.8g}$$

$$\frac{\partial X^*_{(\pm)i}}{\partial u_p^{(m)}}=\pm\varphi_m(z^{(\pm)})\,L^p_{.i3}\pm\mu\varphi_m'(z^{(\pm)})\,\delta_i^p+\frac{\partial\Psi_{i3}^{(\pm)}}{\partial u_p^{(m)}}; \tag{11.9a}$$

$$\frac{\partial X^*_{(\pm)3}}{\partial u_p^{(m)}}=\pm\varphi_m(z^{(\pm)})\,L^p_{.33}\pm(\lambda+2\mu)\,\varphi_m'(z^{(\pm)})\,\delta_3^p+\frac{\partial\Psi_{33}^{(\pm)}}{\partial u_p^{(m)}}; \tag{11.9b}$$

$$\frac{\partial X^*_{(\pm)i}}{\partial(\partial_j u_p^{(m)})}=\pm\mu\varphi_m(z^{(\pm)})\,\delta_3^p\,\delta_j^i+\frac{\partial\Psi_{i3}^{(\pm)}}{\partial(\partial_j u_p^{(m)})}; \tag{11.9c}$$

$$\frac{\partial X^*_{(\pm)3}}{\partial(\partial_j u_p^{(m)})}=\pm\lambda\varphi_m(z^{(\pm)})\,\delta_j^p+\frac{\partial\Psi_{33}^{(\pm)}}{\partial(\partial_j u_p^{(m)})}; \tag{11.9d}$$

$$(i,\ j,\ k=1,\ 2;\ p=1,\ 2,\ 3).$$

Substituting eqs.(11.2a) - (11.3b) and (11.8a) - (11.9d) into eqs.(11.7b), we obtain the equations of motion of an element of the shell in expanded form. Even the linear parts of these equations, however, will be highly cumbersome. The equations of motion can be somewhat simplified by changing the metric on the coordinate axis x^3, and by an appropriate selection of the functions $\varphi_m(z)$. Let us assume that the basic surface of the shell coincides with its middle surface, i.e., that $h_1 = h_2 = h$. Let us put

$$[1-(k_1+k_2)z+k_1k_2z^2]\,dz=d\zeta. \tag{a}$$

Hence, we find

$$\zeta=C+z-\frac{(k_1+k_2)z^2}{2}+\frac{k_1k_2z^3}{3}. \tag{b}$$

Selecting the constant C such that, on variation of z over the interval (-h, +h), the variable ζ varies over the symmetric interval $(-\ell, \ell)$, we obtain

$$\zeta=z+\frac{k_1+k_2}{2}(h^2-z^2)+\frac{k_1k_2z^3}{3}. \tag{c}$$

From eq.(a) results /261

$$dz=ds^3=g_{33}d\zeta, \tag{11.10a}$$

where ds^3 is an element of arc of the third coordinate intersecting the undeformed basic surface at a right angle, and

$$g_{33}=[1-(k_1+k_2)z+k_1k_2z^2]^{-1}. \tag{11.10b}$$

In the last relation, z must be regarded as a function of ζ determined by eq.(c). Of course, for a sufficiently thin shell, at small values of the product zk_i, we can approximately put

$$\zeta\cong z. \tag{11.10c}$$

The introduction of the variable ζ somewhat simplifies eq.(11.7b). This transformation of the coordinate $x^3 = z$ does not affect the form of the notation of the original equations (11.2a) - (11.3b), although the meaning of the variable coefficients of the functions $u_p^{(m)}$ and the first derivatives will differ from their meanings in the original system of coordinates*. In particular, the quantities $\varphi_m(\zeta^{(+)})$ will be functions of the coordinates s^j, i.e., the shell of constant thickness, on introduction of the variable ζ, will be, in a manner of speaking, transformed into a shell of quasi-variable thickness. It is obvious that, at the internal points of the shell, the coordinate ζ, on

* The introduction of a new metric on the third coordinate axis would permit simplifying the notation for certain of the equations considered in Chapter III. In this way, the terms explicitly containing the curvature k_i of the basic surface would be eliminated from these equations.

differentiation, must be regarded as an independent variable. We will not analyze the question of the best method of selecting the functions $\varphi_m(z)$, which now must be replaced in the equations by the functions $\varphi_m(\zeta)$.

In view of the relatively small thickness of the shell, the choice of $\varphi_m(\zeta)$ in the form of a power monomial ζ^m has considerable advantages. This case was essentially considered by us in Chapter III. Here we shall use a different partial selection of the functions $\varphi_m(\zeta)$. Let us put

$$\varphi_{2n} = \cos\frac{n\pi\zeta}{l}; \quad \varphi_{2n+1} = \sin\frac{n\pi\zeta}{l}. \tag{11.11}$$

Consequently,

$$\varphi_0 = 1; \; \varphi_1 = 0; \; \varphi_2 = \cos\frac{\pi\zeta}{l}; \; \varphi_3 = \sin\frac{\pi\zeta}{l}; \; \varphi_4 = \cos\frac{2\pi\zeta}{l} \quad \text{etc.}$$

We recall that the quantity l is a function of the coordinates s^j. The system of functions $\varphi_m(\zeta)$ introduced by eqs.(11.11), is orthogonal over the interval $(-l, l)$. The same property is possessed by the derivatives $\varphi_m'(\zeta)$ and $\varphi_m''(\zeta)$.

We also note the following relations which result from eqs.(11.11): /262

$$\varphi_{2n}' = -\frac{n\pi}{l}\varphi_{2n+1}; \quad \varphi_{2n}'' = -\left(\frac{n\pi}{l}\right)^2\varphi_{2n}; \; \varphi_{2n+1}' = \frac{n\pi}{l}\varphi_{2n};$$

$$\varphi_{2n+1}'' = -\left(\frac{n\pi}{l}\right)^2\varphi_{2n+1}. \tag{11.12}$$

Consider now the equations of motion resulting from eqs.(11.7b). As will be seen from eqs.(11.7b), this system of equations is an infinite system consisting of nonlinear differential equations of the fourth order. It is clear that a direct application of such a system is unpromising. For this reason, we must define the conditions under which this system is resolved into separate subsystems containing a finite number of equations.

The infinite system of equations resulting from eqs.(11.7b) can be decomposed if, in setting up the Euler-Lagrange-Ostrogradskiy equations, we eliminate the necessity of variation of the nonlinear terms and of the terms with the coefficients $M^{*p}_{\cdot\cdot q}$, $N^p_{\cdot q}$ entering into the composition of the body forces ρF^*_q, and also eliminate the variation of the surface forces $X^*_{(\pm)q}$.

In order to eliminate the variation of terms with the coefficients $M^{*p}_{\cdot\cdot q}$ and $N^p_{\cdot q}$ as well as the nonlinear terms, we must replace, in these terms, the functions $u_p^{(m)}$ by expansions of the form

$$u_p^{(m)} = \sum_{\alpha=1}^{\beta} a_{p\alpha}^{(m)} \theta_{p\alpha}^{(m)}, \tag{11.13}$$

where $a_{p\alpha}^{(m)}$ are constant coefficients to be determined, and $\theta_{p\alpha}^{(m)}$ represent a system of functions satisfying the kinematic boundary conditions and the conditions of completeness, ensuring the possibility of approximation of solutions of the equations of the theory of shells by expressions of the form of eq.(11.13)*. The second method of approximate representation of these terms, based on the method of successive approximations, will be given below.

To make it unnecessary to vary the surface forces $X^*_{(\pm)}$, let us employ the following method: Let us resolve the components of the prescribed body forces $X_{(\pm)q}$ into the components $k_{(\pm)q}X_{(\pm)q}$ and $(1 - k_{(\pm)q})X_{(\pm)q}$. The first summand will be regarded as a component of the body force $\delta(\zeta \mp \ell)k_{(\pm)q}X_{(\pm)q}$, where $\delta(\zeta \mp \ell)$ is the delta function. These body forces will be associated with the prescribed forces. Let us also associate with the body forces the /263 terms of the body forces ρF^*_q containing the components $u_p^{(m)}$ expressed by eqs.(11.13). Let us denote the new components of the body forces, for brevity, by the symbol ρR_q, but remember that they depend on the coefficients $a_{p\alpha}^{(m)}$. Then, instead of the functional I expressed by eq.(11.4), we obtain

$$I_1 = \int_0^{t_1} \int_{(S)} 2W \, dS \, dt, \tag{11.14}$$

where, in distinction to eq.(11.5),

$$2W = \int_{-l}^{+l} \left[\sum_{q=1}^{3} (\rho R_q - \rho R_q^*)^2 \right] d\zeta + \sum_{q=1}^{3} [(1 - k_{(\pm)q}) X_{(\pm)q} - X^*_{(\pm)q}]^2. \tag{11.15}$$

where ρR_q^* are components of the body forces which can be determined from eqs.(11.2a) - (11.2b) if the terms entering with opposite signs into the quantities ρR_q are cancelled out from the right-hand side of these equations. Equating to zero in eq.(11.15) the terms containing the surface forces, we find

$$k_{(\pm)q} = 1 - \frac{X^*_{(\pm)q}}{X_{(\pm)q}}. \tag{11.16}$$

Equation (11.16) shows that the coefficients $k_{(\pm)q}$ characterize the accuracy with which the boundary conditions are satisfied on the boundary surfaces of the shell. If these conditions are satisfied exactly on the boundary sur-

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* More detailed information on the conditions which must be satisfied by the functions $\theta_{p\alpha}^{(m)}$ can be found by the reader in manuals on the theory of elasticity, where the Ritz method is set forth.

faces of this shell, then the quantities $k_{(\pm)q}$ will vanish. In accordance with the Kirchhoff-Love hypotheses, the components $X^*_{(\pm)q}$ will vanish and the coefficients $k_{(\cdot)q}$ of the nonvanishing $X_{(\pm)q}$, will be equal to unity.

The conditions of a minimum for the functional I_1 lead to the Euler-Lagrange-Ostrogradskiy system of equations, which in this case will be of the following form:

$$\int_{-l}^{+l}\left\{\sum_{q=1}^{3}\left[(\rho R_q-\rho R_q^*)\frac{\partial(\rho R_q^*)}{\partial u_p^{(m)}}-\right.\right.$$

$$-\sum_{j=1}^{2}\frac{\partial}{\partial s^j}\left[(\rho R_q-\rho R_q^*)\frac{\partial(\rho R_q^*)}{\partial(\partial_j u_p^{(m)})}\right]+$$

$$+\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{\partial^2}{\partial s^j\partial s^k}\left[(\rho R_q-\rho R_q^*)\frac{\partial(\rho R_q^*)}{\partial(\partial_j\partial_k u_p^{(m)})}\right]+$$

$$\left.\left.+\frac{\partial^2}{\partial t^2}\left[(\rho R_q-\rho R_q^*)\frac{\partial(\rho R_q^*)}{\partial\left(\frac{\partial^2 u_p^{(m)}}{\partial t^2}\right)}\right]\right]\right\}d\zeta=0$$

$$(p=1,\ 2,\ 3;\ m=0,\ 1,\ 2,\ \ldots\ n). \tag{11.17}$$

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In addition to eqs.(11.17), we must also bear in mind the conditions determining the coefficients $a_{p\alpha}^{(m)}$ in eqs.(11.13). These conditions have the following form, well known from the Ritz method:

$$\frac{\partial I_1}{\partial a_{p\alpha}^{(m)}}=0. \tag{11.18}$$

Thus, the complete syst[illegible] of equations of the theory of shells now consists of eqs.(11.16), (11.17), and (11.18). This system combines the classical Euler-Lagrange-Ostrogradskiy equations with the equations resulting from the Ritz method. Let us now consider eqs.(11.17). We shall first introduce an abbreviated notation for several differential operators:

$$\rho\frac{\partial^2 u_i^{(2m)}}{\partial t^2}-\mu\sum_{k=1}^{2}\partial_k^2 u_i^{(2m)}-(\lambda+\mu)\sum_{k=1}^{2}\partial_i\partial_k u_k^{(2m)}-$$

$$-(\lambda+\mu)\left(\frac{m\pi}{l}\right)\partial_i u_3^{(2m+1)}+\mu\left(\frac{m\pi}{l}\right)^2 u_i^{(2m)}=P_i^{(2m)}(\vec{u});$$

(11.19a)

$$\rho\frac{\partial^2 u_i^{(2m+1)}}{\partial t^2}-\mu\sum_{k=1}^{2}\partial_k^2 u_i^{(2m+1)}-(\lambda+\mu)\sum_{k=1}^{2}\partial_i\partial_k u_k^{(2m+1)}+$$

$$+(\lambda+\mu)\left(\frac{m\pi}{l}\right)\partial_i u_3^{(2m)}+\mu\left(\frac{m\pi}{l}\right)^2 u_i^{(2m+1)}=P_i^{(2m+1)}(\vec{u});$$

(11.19b)

$$\rho\frac{\partial^2 u_3^{(2m)}}{\partial t^2}-\mu\sum_{k=1}^{2}\partial_k^2 u_3^{(2m)}-(\lambda+\mu)\left(\frac{m\pi}{l}\right)\sum_{k=1}^{2}\partial_k u_k^{(2m+1)}+$$

$$+(\lambda+2\mu)\left(\frac{m\pi}{l}\right)^2 u_3^{(2m)}=P_3^{(2m)}(\vec{u});$$

(11.19c)

$$\rho\frac{\partial^2 u_3^{(2m+1)}}{\partial t^2}-\mu\sum_{k=1}^{2}\partial_k^2 u_3^{(2m+1)}+(\lambda+\mu)\left(\frac{m\pi}{l}\right)\sum_{k=1}^{2}\partial_k u_k^{(2m+1)}+$$

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$$+(\lambda+2\mu)\left(\frac{m\pi}{l}\right)^2 u_3^{(2m+1)}=P_3^{(2m+1)}(\vec{u}).$$

(11.19d)

The operators $P_q^{(n)}$ ($q = 1, 2, 3$) depend on the curvature k_i of the basic surface, since these curvatures enter into the parameter ℓ. However, this dependence can be termed weak, especially for the case of thin shells with the ratio $2h : R_{max} \leq 0.01$. The operators $P_q^{(n)}$ are apparently close to the operators describing the linear stress-strain state of a given plate.

Further, let us denote the operators depending on the curvature of the basic surface of the shell and on the coordinate ζ as follows:

$$M^{kp}_{..q}\partial_k u_p^{(2m)}+N^p_{.q}u_p^{(2m)}+\left(\frac{m\pi}{l}\right)M^{3p}_{..q}u_p^{(2m+1)}=Q_q^{(2m)}(\vec{u});\qquad(11.20a)$$

$$M^{kp}_{..q}\partial_k u_p^{(2m+1)}+N^p_{.q}u_p^{(2m+1)}-\left(\frac{m\pi}{l}\right)M^{3p}_{..q}u_p^{(2m)}=Q_q^{(2m+1)}(\vec{u})$$

$$(q=1,\ 2,\ 3).\qquad(11.20b)$$

Then, bearing in mind eqs.(11.12), we find

$$\rho R_q^*=\sum_{m=0}^{N}[\varphi_{2m}P_q^{(2m)}+\varphi_{2m+1}P_q^{(2m+1)}];\qquad(11.21a)$$

$$\rho R_q = \rho F_q - \sum_{m=0}^{N} [\varphi_{2m} Q_q^{(2m)} + \varphi_{2m+1} Q_q^{(2m+1)}] + \Psi_q +$$

$$+ \delta(\zeta - l)\, k_{(+)q} X_{(+)q} + \delta(\zeta - l)\, k_{(-)q} X_{(-)q}$$
$$(q = 1,\ 2,\ 3). \qquad (11.21b)$$

Let us continue our consideration of the quantities entering into eqs.(11.17). Again bearing in mind eqs.(11.12), we find

$$\frac{\partial(\rho R_i^*)}{\partial u_p^{(2n)}} = \mu\, \varphi_{2n} \left(\frac{n\pi}{l}\right)^2 \delta_i^p; \quad \frac{\partial(\rho R_i^*)}{\partial u_p^{(2n+1)}} = \mu \varphi_{2n+1} \left(\frac{n\pi}{l}\right)^2 \delta_i^p; \qquad (11.22a)$$

$$\frac{\partial(\rho R_3^*)}{\partial u_p^{(2n)}} = (\lambda + 2\mu)\, \varphi_{2n} \left(\frac{n\pi}{l}\right)^2 \delta_3^p; \quad \frac{\partial(\rho R_3^*)}{\partial u_p^{(2n+1)}} = (\lambda + 2\mu)\, \varphi_{2n+1} \left(\frac{n\pi}{l}\right)^2 \delta_3^p; \qquad (11.22b)$$

$$\frac{\partial(\rho R_i^*)}{\partial(\partial_j u_p^{(2n)})} = (\lambda + \mu) \left(\frac{n\pi}{l}\right) \varphi_{2n+1}\, \delta_j^i \delta_3^p; \quad \frac{\partial(\rho R_i^*)}{\partial(\partial_j u_p^{(2n+1)})} =$$
$$= -(\lambda + \mu) \left(\frac{n\pi}{l}\right) \varphi_{2n} \delta_j^i\, \delta_3^p; \qquad (11.22c)$$

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$$\frac{\partial(\rho R_3^*)}{\partial(\partial_j u_p^{(2n)})} = (\lambda + \mu) \left(\frac{n\pi}{l}\right) \varphi_{2n+1}\, \delta_j^k \delta_k^p; \quad \frac{\partial(\rho R_3^*)}{\partial(\partial_j u_p^{(2n+1)})} =$$
$$= -(\lambda + \mu) \left(\frac{n\pi}{l}\right) \varphi_{2n}\, \delta_j^k\, \delta_k^p; \qquad (11.22d)$$

$$\frac{\partial(\rho R_i^*)}{\partial(\partial_j \partial_k u_p^{(n)})} = -\varphi_n [\mu \delta_i^p\, \delta_j^k + (\lambda + \mu)\, \delta_j^i \delta_k^p]; \qquad (11.22e)$$

$$\frac{\partial(\rho R_3^*)}{\partial(\partial_j \partial_k u_p^{(n)})} = -\mu \varphi_n \delta_3^p\, \delta_j^k; \qquad (11.22f)$$

$$\frac{\partial(\rho R_q^*)}{\partial\left(\dfrac{\partial^2 u_p^{(n)}}{\partial t^2}\right)} = \rho \varphi_n \delta_q^p \qquad (11.22g)$$

$$(i,\ j,\ k = 1,\ 2;\ p,\ q = 1,\ 2,\ 3).$$

Equations (11.21a) - (11.22g) yield the meaning of the abbreviated notation [eq.(11.17)] for the system of equations of motion of an element of the shell. For this, it is sufficient to give to the index p the values 1, 2, 3, and to take m successively equal to 2n and 2h + 1. In this way, we obtain the system of equations

$$P_q^{(m)}(\vec{v}) = 0$$
$$(m = 0,\ 1,\ 2, \ldots,\ N). \tag{11.23}$$

where $P_q^{(m)}$ are the operators whose meaning is given by eqs.(11.19a) - (11.19d). The components of the vector $\vec{v}$ are determined in the following manner:

$$V_r^{(h)} = P_r^{(h)}(\vec{u}) - \frac{1}{l}\int_{-l}^{+l} \rho R_r \varphi_h d\zeta \tag{11.24}$$
$$(h = 0,\ 1,\ 2, \ldots;\ r = 1,\ 2,\ 3).$$

Equations (11.23) constitute, at $N \to \infty$, an infinite system of fourth-order equations. If no components of the multi-dimensional vector $\vec{v}$ enter in the function R_r, then the system of equations (11.23) is resolved into autonomous subsystems, each containing six equations. This resolution will take place directly only in problems of the mechanics of plates or shells with zero curvatures k_i of the basic surface.

In the remaining cases, no decomposition takes place. It is therefore necessary to determine the components of the vector $\vec{u}$ entering into the quantities ρR_r by eqs.(11.13). In this case, as already noted, eqs.(11.23) will contain indeterminate coefficients $a_{p\alpha}^{(m)}$. To determine these quantities we /267
must use eqs.(11.18), which are algebraic equations that are linear for the linear statement of the problem and nonlinear in the general case. We will not give the expanded form of these equations. We call attention only to one of the features of the method developed by us.

This method is based on the combined use of eqs.(11.23) - (11.24) determining the wanted functions as solutions of a certain boundary problem, and on eqs.(11.18) which, when taken together with eqs.(11.13), yield the approximate analytic form of the solution. The method of combination of eqs.(11.23) - (11.24) and eq.(11.18) together with eq.(11.13) depends on the scope of the problem of mechanics involved. In any particular approach to investigation of the problem, however, we will obtain an approximate representation of the motion of a shell element, which differs from solutions that are analogous but are obtained from other equations in that, in this case, the necessary conditions of the minimum of the quadratic deviation from the solutions of the three-dimensional theory of elasticity will be satisfied.

We note finally that the condition of minimum of the functional I_1 determined by eq.(11.14), actually coincides with the Gauss principle of least constraint Z of the system, if the constraint Z is averaged over the time interval $(0,\ t_1)$.

Section 12. Boundary Conditions. Various Versions of the Solution of the General Problem of the Dynamics of Shells. Initial Conditions

1. Remarks on Boundary Conditions

Let us consider the boundary conditions that terminate the statement of the problem and briefly discuss the various versions of its solution.

To obtain a system of boundary conditions, let us find, on the contour surface of the shell, the components of the displacement vector and stress vector resulting from the approximate representations (11.1). Then, considering the shell as a three-dimensional body, let us set up the boundary conditions on the contour surface of the shell in accordance with the statement of the three-dimensional problems of the theory of elasticity considered in Chapter II. Finally, using the method of least squares, let us require that the quantities resulting from the approximate representations (11.1) shall satisfy, on the contour surface of the shell, the requirement of the least-square deviation from the corresponding functions prescribed on the contour surface in the formulation of the three-dimensional boundary problem of the theory of elasticity.

This program requires consideration of the following functionals: /268

$$I_2=\int_0^{t_1}\int_{(C_1)}\int_{-l}^{+l}\left\{\sum_{q=1}^{3}(u_q-u_q^*)^2\right\}d\zeta\,ds\,dt, \tag{12.1a}$$

$$I_3=\int_0^{t_1}\int_{(C-C_1)}\int_{-l}^{+l}\left\{\sum_{q=1}^{3}(f_q-f_q^*)^2\right\}d\zeta\,ds\,dt. \tag{12.1b}$$

where C is the contour of the middle surface of the shell, C_1 is that part of the contour on which the displacements are prescribed, u_q and f_q are the components of the displacement and stress vectors prescribed on the respective parts of the contour surface, and u^*_q and f^*_q are the approximate expressions of these components determined by eqs.(11.1). For definiteness we may assume that the functions $\varphi_m(\zeta)$ are expressed by eqs.(11.11). Let us denote

$$2W_2=\int_{-l}^{+l}\left\{\sum_{q=1}^{3}(u_q-u_q^*)^2\right\}\Bigg|_{C_1}d\zeta;\quad 2W_3=\int_{-l}^{+l}\left\{\sum_{q=1}^{3}(f_q-f_q^*)^2\right\}\Bigg|_{C-C_1}d\zeta. \tag{12.2}$$

The conditions of a minimum for the functionals I_2 and I_3 lead to the following conditions of the middle surface of the shell:

$$\frac{\partial W_2}{\partial u_p^{(m)}}\Bigg|_{C_1}=0;\quad \frac{\partial W_3}{\partial u_p^{(m)}}\Bigg|_{C-C_1}-\sum_{j=1}^{2}\frac{\partial}{\partial s^j}\frac{\partial W_3}{\partial(\partial_j u_p^{(m)})}\Bigg|_{C-C_1}=0, \tag{12.3a}$$

or

$$\int_{-l}^{+l}\left\{\sum_{q=1}^{3}(u_q-u_q^*)\frac{\partial u_q^*}{\partial u_p^{(m)}}\right|_{C_1}d\zeta=0; \tag{12.3b}$$

$$\int_{-l}^{+l}\left\{\sum_{q=1}^{3}\left[(f_q-f_q^*)\frac{\partial f_q^*}{\partial u_p^{(m)}}-\sum_{j=1}^{2}\frac{\partial}{\partial s^j}\left[(f_q-f_q^*)\frac{\partial f_q^*}{\partial(\partial_j u_p^{(m)})}\right]\right]\right\}\Bigg|_{C-C_1}d\zeta=0$$

$$(p=1,\ 2,\ 3;\ m=0,\ 1,\ 2,\ldots,\ N). \tag{12.3c}$$

We recall that, on the contour C, the derivatives $\frac{\partial}{\partial s^j}$ must be expressed in terms of the derivatives along the tangent and along the principal normal to the contour C. We shall not consider the conditions (12.3b) - (12.3c) in the expanded form and confine ourselves to brief remarks on the statement of the boundary problem of the dynamics of shells considered here.

a) We cannot directly assert that the boundary conditions (12.3b)-(12.3c) are natural for the variational problem considered in Sect.11, at least not /269 with respect to the method by which they were obtained. We must, therefore, define their connection with the natural boundary conditions. We will do this later in the text.

b) Naturally the question arises as to the existence and uniqueness of a solution of the boundary problem under consideration.

2. On the Existence and Uniqueness of Solutions of the Boundary Problem Posed

It is well known that theorems for the uniqueness of solutions of linear static and dynamic three-dimensional problems of the theory of elasticity for finite regions have long since been proved*. Theorems of the existence of solutions have been proved for three-dimensional linear problems of the statics of an elastic problem and also for a number of problems of dynamics**.

Since we are here investigating only the results of the approximation of the equations of the three-dimensional internal problem of the theory of elasticity, leading to equations of the elastodynamics of shells, we may in advance assume with considerable certainty that these theorems on the existence and uniqueness of solutions can also be extended to the boundary problems under consideration. Of course, this is merely a working hypothesis.

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* Cf. for instance, E.Trefftz, Mathematical Theory of Elasticity, ONTI, 1934

** See preceding footnote and also V.D.Kupdraze, Boundary Problems of the Theory of Vibrations and Integral Equations, Gostekhizdat, 1950

We will not further discuss the proofs of these theorems with respect to the theory of shells; confining ourselves to the following remark: If we do not require satisfaction of the necessary conditions (12.3b) - (12.3c) for the extremum of the functionals I_2 and I_3, then it can be asserted that there exists a solution of eqs.(11.18) and (11.23) for which the sum $I_2 + I_3$ will have a minimum. The question of the uniqueness of such a solution remains open.

3. Natural Boundary Conditions

There is no difficulty in determining the natural boundary conditions for the problem of the extremum of the functional I_1, expressed by eq.(11.14). However, we will not investigate these conditions, recalling that the boundary conditions for the three-dimensional problem of the theory of elasticity considered in Chapt.II are natural*. The conditions (12.3b) - (12.3c) found by us are the results of the requirement of least-square deviations of the solutions of the problem of the elastodynamics of shells from the natural boundary conditions of the three-dimensional problem of the theory of elasticity on the /270 contour surface of the shell. This requirement is in agreement with the fundamental principle of constructing the equations of the elastodynamics of shells as applied in Sect.11. In this connection, we may consider the conditions (12.3b) - (12.3c) as natural conditions for the extremum of the functional I_1 in an extended sense, even if they do not coincide with the natural boundary conditions of the variational problem.

In conclusion, let us discuss possible versions of the solution of the system of equations (11.16), (11.18), (11.23), (11.24) with the boundary conditions (12.3b) - (12.3c).

First let us concentrate on the system of functions $k_{(\pm)q}$. Although the system of equations constructed by us is complete, i.e., the number of equations is equal to the number of functions sought, it is not advisable to attempt an "exact" determination of the quantities $k_{(\pm)q}$ from these equations. The iteration method should be used. For the beginning, let us put the functions $k_{(\pm)q}$ equal to zero or unity, zero corresponding to the exact satisfaction of the boundary conditions on the boundary surfaces of the shell, and unity to the application of the Kirchhoff-Love hypotheses. Then, adopting one or the other method, let us proceed to the solution of the boundary problem.

In Section 11 we mentioned one of the procedures for this solution, based on eqs.(11.13). A different version of the solution is possible, permitting us to exclude from consideration the approximate expressions for the components $u_j^{(*)}$ and, consequently, eqs.(11.18). This version is based on an iteration process.

Assigning to the coefficients $k_{(\pm)q}$ definite values and rejecting in the quasi-body forces ρR_p all terms depending on the quantity $u_j^{(*)}$, let us solve the boundary problem of the shell theory. Then, from eqs.(11.16), we find the corrected values of $k_{(\pm)q}$ and introduce into ρR_q the terms depending on the

* Cf., for instance, V.I.Smirnov, Course in Higher Mathematics, Vol.IV, p.295, Gostekhizdat, 1951

quantities $u_p^{(0)}$ found as a result of the original approximation. The process is then repeated. Evidently, we can expect positive results when this method is applied to the calculation of plane shells, since most of the terms rejected in obtaining the original approximation depend on the curvature of the middle surface of the shell.

As in the preceding Chapter, we note the correlation of the solution of the elastodynamic equations of the theory of shells, found by the method under study, with the general methods of the mathematical theory of elasticity. When determining the components of the displacement vector by the approximate formulas (11.1), we are evidently able to find the components of the strain tensor, and, from Hooke's law, the components of the stress tensor.

The Saint-Venant compatibility conditions will be satisfied. The equations of motion on the boundary conditions will be approximately satisfied. Here the satisfaction of the boundary conditions on the boundary surfaces of the shell can be improved by selecting the functions $k_{(\pm)q}$. The introduction of the functions $k_{(\pm)q}$ is one of the features of the proposed method. /271

We shall now take up a question not yet discussed. The equations obtained in Sect.11 differ from (III, 24.27 - 24.29) found from the general equation of dynamics, in being of a higher order. This results in certain complications in stating the initial conditions. It is natural to use here a method based on the approximate satisfaction of the initial conditions, starting from the requirements of the least-square deviation of the approximation functions from the functions describing these conditions.

Let us assume in accordance with (II, 8.1a - 8.1b) that, at the initial time t_0, the components of the displacement vector u_{i0} and of the velocity vector $\dot{u}_0$ are assigned as functions of the coordinates of a point of the shell. If these functions are differentiable, then, by differentiating them we shall find the field of the initial rate of deformation.

In certain problems of dynamics, the functions u_{i0} and u_{i0} are not differentiable. An example is given by the initial values of the longitudinal velocities in a rod under longitudinal impact. If the functions u_{i0} and $\dot{u}_{i0}$ are not differentiable, the initial fields of deformation and of the rate of deformation must be independently prescribed.

To set up the initial conditions of the problem, let us consider the four functionals:

$$I_1 = \int\limits_{(S)} \int\limits_{-l}^{+l} [u_{i0} - u_{i0}^*]^2 \, d\zeta \, dS; \quad I_2 = \int\limits_{(S)} \int\limits_{-l}^{+l} [\dot{u}_{i0} - \dot{u}_{i0}^*]^2 \, d\zeta \, dS; \tag{12.4a}$$

$$I_3 = \int\limits_{t_0}^{t_0+\varepsilon} \int\limits_{(S)} \int\limits_{-l}^{+l} [L - L^*]^2 \, d\zeta \, dS \, dt; \quad I_4 = \int\limits_{t_0}^{t_0+\varepsilon} \int\limits_{(S)} \int\limits_{-l}^{+l} [\dot{L} - \dot{L}^*]^2 \, d\zeta \, dS \, dt. \tag{12.4b}$$

where ϵ is an arbitrary small time interval. The Lagrange function $\dot{L}$ is established from the components of the velocity vector of a shell element, just as the function L is established from the components of the displacement vector. The other notation is familiar from the preceding discussion.

We will define the initial conditions, starting from the conditions that the functionals I_1, I_2, I_3 and I_4 shall be minimum. The physical meaning of this requirement is obvious. As for the functional I_4, it contains the "energy of acceleration" which enters into the Gauss principle of least constraint, and the rate-of-strain energy. Thus, this functional is connected with the quantities characterizing the minimum properties of the accelerations of the actual motion of the system. It would also be possible to introduce direct- /272
ly the Gaussian constraint Z of the system, but this would make it impossible to use the analytic apparatus employed above.

The requirement for minimizing the functionals I_1, I_2, I_3, and I_4 is obviously equivalent to the optimum simulation of the initial mechanical state of the shell by the quantities resulting from the approximate equations (11.1). If we introduce the notation

$$W_{10}=\int_{-l}^{+l}[u_{i0}-u_{i0}^{*}]^2\,d\zeta,\quad W_{20}=\int_{-l}^{+l}[\dot{u}_{i0}-\dot{u}_{i0}^{*}]^2\,d\zeta,\tag{12.5}$$

$$W_{30}=\int_{-l}^{+l}[L-L^{*}]^2\,d\zeta,\quad W_{40}=\int_{-l}^{+l}[\dot{L}-\dot{L}^{*}]^2\,d\zeta,$$

then the necessary conditions for the functionals I_1, I_2, I_3 and I_4 to be minimum will take the following form:

$$\frac{\partial W_{10}}{\partial u_{p0}^{(m)}}=0;\quad \frac{\partial W_{20}}{\partial \dot{u}_{p0}^{(m)}}=0;\tag{12.6a}$$

$$\left[\frac{\partial W_{30}}{\partial u_p^{(m)}}-\sum_{j=1}^{2}\frac{\partial}{\partial s^j}\frac{\partial W_{30}}{\partial(\partial_j u_p^{(m)})}-\frac{\partial}{\partial t}\frac{\partial W_{30}}{\partial \dot{u}_p^{(m)}}\right]_{t=t_0}=0;\tag{12.6b}$$

$$\left[\frac{\partial W_{40}}{\partial \dot{u}_p^{(m)}}-\sum_{j=1}^{2}\frac{\partial}{\partial s^j}\frac{\partial W_{40}}{\partial(\partial_j \dot{u}_p^{(m)})}-\frac{\partial}{\partial t}\frac{\partial W_{40}}{\partial \ddot{u}_p^{(m)}}\right]_{t=t_0}=0\tag{12.6c}$$

$$(p=1,\ 2,\ 3;\ m=0\ \ 1,\ 2,\ldots).$$

Equations (12.6a) permit a direct determination of the initial values $u_{p0}^{(m)}$ and $\dot{u}_{p0}^{(m)}$. Substituting these values into eqs.(12.6b), we obtain the initial values $\ddot{u}_{p0}^{(m)}$ and then, passing to eqs.(12.6c), the initial values $\dddot{u}_p^{(m)}$. Of

course, there may be other approaches to the determination of the extended system of initial conditions.

In this Chapter, we have considered the method of linear approximation of the components of the finite-deformation tensor. Such an approximation permits eliminating only a part of the nonlinear terms from the Lamé equations, but does not exhaust the problem of linearization, since other sources of nonlinearity still remain, including the Christoffel symbols $\{^{\,i}_{jk}\}$ of the Lagrangian coordinates of a medium with extensive deformations and nonlinearity in the boundary conditions. For this reason, we must apply the method of linear approximation to the quasi-body forces containing nonlinear terms entering into the Lamé equations. The complexity of the analytic expressions for these quasi-body forces, however, prevents us for the time being from developing the method of linear approximation in a general form. /273

Section 13. Approximate Methods of Investigating the Equilibrium and Oscillations of Shells as Discrete-Continuum Systems*

In concluding our discussion of the problem complex connected with the general problem of constructing a mechanical system closely related, according to certain criteria, to some prescribed system, let us briefly discuss the method of reducing the problems of the theory of shells to problems of the study of motion of systems with a finite number of degrees of freedom. We shall call such a system a discrete-continuous system.

The concept of "discrete-continuous system" was introduced into the theory of shells by V.Z.Vlasov (Bibl.3b). In his terminology such a system is a thin-walled elastic two-dimensional system possessing a finite number of degrees of freedom along one of the coordinates and an infinitely great number along the other.

We will also use this term in what follows, but shall give it a different meaning. By the term "discrete-continuous system" we shall understand a continuous medium whose dynamic state is approximately determined by a system of functions of time related to a discrete set of points on the basic surface of the shell. Whatever other aspects of the concept of discrete-continuous systems are possible, will not be considered here.

The investigation in the final part of this Chapter proposes to give a method of reducing the problem of deriving quantities that determine the stress-strain state of a shell to the solution of finite systems of algebraic equations if the shell is in equilibrium, and to the solution of a system of ordinary differential equations of second or higher order if vibrations of the shell are to be studied.

The principle analytic approach to the construction of equations describing the state of a shell as a discrete-continuous system is the use of inter-

* The substance of this and the following Sections was presented to the All-Union Conference on the Theory of Plates and Shells held at Kazan in 1960. Cf., the author's paper in the Transactions of the Conference.

polation formulas that express the values of the functions sought in terms of a discrete set of their values at the nodes of a certain net*. Consequently, here, too, as in the earlier part of this Chapter, we intend to use one of /274 the methods of approximation functions.

The choice of such a method is arbitrary. We have not investigated the comparative effectiveness of the various methods of approximation functions in application to the basic problem, the replacement of the shell by a system with a finite number of degrees of freedom.

Section 14. The Fundamental Discrete System of Unknowns

To construct the discrete-continuous system replacing the shell, one of the above-considered reduction methods must be used.

We shall apply the methods studied at the beginning of Chapter III. This method is closest to the method based on the Kirchhoff-Love hypotheses. Our results discussed below can, therefore, be extended without fundamental complications to the classical theory of shells.

As shown previously (III, Sect.5), the three-dimensional problem of the theory of elasticity can be reduced to a determination of six functions of a point of the basic surface of the shell. These functions are the displacement vector components u_i of a point of the basic surface of the shell, and the covariant derivatives $\nabla_3 u_i$.

Assume that the values of the six functions, determining the state of the shell, are known at the nodes of some net on the basic surface. Then the value of these functions at the intermediate points can be determined by one of the interpolation formulas. This permits us to express the strains and, by means of Hooke's law, the stresses at an arbitrary point of the shell in terms of the values of unknowns at the nodes of the net.

The interpolation method involves the relative order of the terms retained on reduction of the three-dimensional problem of the theory of elasticity to a two-dimensional problem in expansions of the form of (III, 4.2), (III, 4,5a), (III, 4.5b) and subsequent relations resulting from those enumerated.

Making use of eqs.(II,2.11) and expansions of the form of (III, 4.2), /275 let us consider, for example, elementary and roughly approximate representa-**

* We recall that the use of interpolation formulas was given in the theory of shells by I.Ya.Shtayerman in his work "On the Application of Interpolation Methods to the Approximate Integration of the Differential Equations of Equilibrium of Elastic Shells", Visti KPI, Vol.2, 1927 and in the problems of structural mechanics, by N.V.Kornoukhov in his paper "An Interpolation-Iteration Method of Solving the Differential Equations of Strength and Stability of Prismatic Rods", Sbornik trudov Inst. stroit. mekhan AN UkrSSR, Vol.11, 1949

** This approximation corresponds to the accuracy of determination of the strain tensor components adopted in the classical theory of shells.

tions of the components of the strain tensor,

$$2D_{jk}^{(z)} \cong \nabla_j u_k + \nabla_k u_j + \nabla_j u_3 \nabla_k u^3 + z\,[\nabla_j \vartheta_k + \nabla_k \vartheta_j + \\ + \nabla_j \vartheta_3 \nabla_k u^3 + \nabla_j u_3 \nabla_k \vartheta^3] \qquad (i,\ k = 1,\ 2), \tag{14.1a}$$

$$2D_{p3}^{(z)} = \nabla_p u_3 + \vartheta_p \qquad (p = 1,\ 2,\ 3). \tag{14.1b}$$

where

$$\vartheta_j = \nabla_3 u_j\,|_{z=0}. \tag{14.2}$$

The meaning of the other symbols has been given in Chapter III.

In setting up eqs.(14.1a) - (14.1b), we retained in the nonlinear part of the strain tensor components only those terms with the greatest significance according to the well-known postulates that can be traced back to the investigations by T.Karman.

The right-hand sides of eqs.(14.1a) - (14.1b) contain first-order derivatives with respect to the coordinates x^i (i = 1, 2). For this reason, remaining at least within the limits of accuracy adopted in the net method, let us apply the following interpolation methods: Let us cover the basic surface with a triangulation net, and within each triangle let us interpolate the unknown functions by linear functions of the coordinates of the basic surface, taking values equal to the values of the unknown functions at the vertices of the triangle.

Consider for example the triangle M_1(p, q), M_2(p + 1, q), M_3(p, q + 1). Here, p and q are the numbers of the nodes of the net on the coordinate lines. The component of displacement u_j within the triangle M_1, M_2, M_3 will be expressed by the equation

$$\begin{vmatrix} u_j - u_j(M_1) & x^1 - x^1(M_1) & x^2 - x^2(M_1) \\ u_j(M_2) - u_j(M_1) & x^1(M_2) - x^1(M_1) & x^2(M_2) - x^2(M_1) \\ u_j(M_3) - u_j(M_1) & x^1(M_3) - x^1(M_1) & x^2(M_3) - x^2(M_1) \end{vmatrix} = 0, \tag{14.3a}$$

or

$$u_j = (a_{j1}x^1 + b_{j1}x^2 + c_{j1})\,u_j(M_1) + (a_{j2}x^1 + b_{j2}x^2 + c_{j2})\,u_j(M_2) + \\ + (a_{j3}x^1 + b_{j3}x^2 + c_{j3})\,u_j(M_3). \tag{14.3b}$$

The coefficients of the linear trinomials which are the factors of the components $u_j(M_p)$ in eq.(14.3b) can be found from a comparison of eqs.(14.3a) with eq.(14.3b). We will not give the expressions for these components. The

functions ϑ_j within the triangle $M_1M_2M_3$ are similarly determined.

The quantities $u_j(M_p)$ and $\vartheta_j(M_p)$ in the problems of dynamics are functions of the time t. In problems of statics, they do not depend on the time. /276

Approximating the displacements $u_j^{(z)}$ by linear functions of z, we complete the construction of the quantities that approximately characterize the state of a prismatic element of the shell resting on the triangle $M_1M_2M_3$.

Making use of eqs.(14.1a) - (14.1b), we can approximately determine the strain-stress state of the shell in the prismatic element resting on the triangle $M_1M_2M_3$. In relative accuracy, this determination corresponds (for example) to the accuracy with which the stress-strain state is determined for a one-dimensional rod in longitudinal vibration, if the rod is replaced by a system of concentrated masses connected by weightless springs. The difference is primarily that, in the rod, when this method of approximate solution of the problem of longitudinal vibrations is used, the single component of the strain tensor has discontinuities at the points at which the mass is concentrated, while in the case under consideration the faces of the prismatic elements will be surfaces of separation of the strain tensor components. The six principal quantities, however, will retain their continuity on these surfaces.

Further refinements will lead to an increase in the number of terms retained in expansions of the form of eqs.(14.1a) - (14.1b) and the consequences, as we know from Chapter III, will introduce into the expansion derivatives of second and higher order with respect to the coordinates of the basic surface, so that the linear approximations of the form of eqs.(14.3a) - (14.3b) will become insufficient. We must take recourse to interpolation formulas in the form of polynomials of the coordinates x^j ($j = 1, 2$) of the second, third, and higher orders. This will complicate the base region of approximation.

In an approximation by linear trinomials, such a region, as already mentioned, is a triangle. For an approximation by polynomials of the second degree we may, for instance, use a "double" triangle $M_1(p, q)$, $M_2\left(p + \frac{1}{2}, q\right)$, $M_3(p + 1, q)$ $M_4\left(p, q + \frac{1}{2}\right)$, $M_5(p, q + 1)$, $M_6\left(p + \frac{1}{2}, q + \frac{1}{2}\right)$, to approximate the polynomials of the third-degree "triple" triangle with an additional internal point, etc.. The "fractional" numbering of the interpolation nodes indicates the position of an auxiliary node between the nodes of the main region, which remains a triangle with integers used in numbering its nodes. For example, the node $M_2\left(p + \frac{1}{2}, q\right)$ lies on the straight line joining the /277 nodes $M_1(p, q)$ and $M_3(p + 1, q)$. All these cases lead to approximation formulas for the six principal functions that are linear with respect to the values of these functions at the nodes of the net.

The coefficients of the values of the principal unknowns at the nodes of a net are polynomials of the coordinates x^i ($i = 1, 2$), equal to unity at the

respective node and equal to zero at all other nodes. We have

$$u_j=\sum_{(p,\,q)} u_j(p,\ q)\,\varphi_{pq}(x^i); \tag{14.4a}$$

$$\vartheta_j=\sum_{(p,\,q)} \vartheta_j(p,\ q)\,\psi_{pq}(x^i). \tag{14.4b}$$

The sums are extended to the base regions of approximation indicated above. Let us denote the coordinates of the nodes by x^1_p and x^2_q. Then

$$\varphi_{pq}(x^1_p,\ x^2_q)=1;\quad \varphi_{pq}(x^1_r,\ x^2_s)=0 \qquad (r,\ s\neq p,\ q); \tag{14.5a}$$

$$\psi_{pq}(x^1_p,\ x^2_q)=1;\quad \psi_{pq}(x^1_r,\ x^2_s)=0 \qquad (r,\ s\neq p,\ q). \tag{14.5b}$$

where, as mentioned above, φ_{pq} and ψ_{pq} are polynomials of the coordinates x^i of the basic surface of the shell.

The expressions (14.4a) - (14.4b), as well as the reduction formulas considered in Chapter III, yield approximate expressions for the potential and kinetic energy of the shell and make it possible to construct the Lagrange function L*. The generalized coordinates here will be the quantities $u_j(p,\ q)$ and $\vartheta_j(p,\ q)$. Among these generalized coordinates, however, there may also be redundant coordinates, since the boundary conditions of the problem impose, on the quantities $u_j(p,\ q)$ and $\vartheta_j(p,\ q)$, restrictions which are analytically expressed by the equations of geometric connectivity, and in the general case, of kinematic connectivity. Let us consider this question in greater detail.

Section 15. Boundary Conditions and the Equations of Connectivity. Initial Conditions

In considering the boundary conditions we shall start from the concepts of the three-dimensional stress-strain state of the shell, as adopted in Chapter III.

The various boundary conditions introduce no additional complications into the solution of the problem by this method. Kinematic, kinetic and mixed boundary conditions may be prescribed on the contour surface. These conditions lead to equations of linear and nonlinear geometric and kinematic connectiv- /278 ity.

Without going into detail, let us consider the cases of the principal boundary conditions prescribed on the contour surface of the shell.

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* Here L is the Lagrange function for the shell as a whole, rather than the density of the Lagrange function considered above, and is termed for brevity the "Lagrange function".

1. First Boundary Condition

On the contour surface C, let the displacements * be given:

$$(u_i^{(z)})_C = \varphi_i(x^j, z, t) \qquad (i = 1, 2, 3; j = 1, 2). \tag{15.1}$$

Making use of the notation adopted in Chapter III, and setting

$$u_i^{(1)} = \vartheta_i, \tag{15.2}$$

we find, from (III, 6.4a) - (III, 6.4b), equations valid over the entire basic surface, including its contour C.

$$u_j^{(z)} = u_j + z\vartheta_j + \frac{1}{2} z^2 \left[L_j \vartheta_3 + L_{j.}^{s} u_s + M u_j - \frac{\rho}{\mu} F_j \right] + \dots \tag{15.3a}$$

$$u_3^{(z)} = u_3 + z\vartheta_3 + \frac{1}{2} z^2 \left[N_{3.}^{;s} \vartheta_s + \frac{\mu}{\lambda + 2\mu} M u_3 - \frac{\rho}{\lambda + 2\mu} F_3 \right] + \dots$$

$$(j, s = 1, 2). \tag{15.3b}$$

Again making use of the expansions (III, 13.3) and confining ourselves to three terms of the expansions on the right-hand sides of eqs.(15.3a) - (15.3b) we obtain

$$(u_i)_C = \varphi_i(x^j, 0, t); \; (\vartheta_i)_C = \varphi_i^{(1)}(x^j, 0, t)$$

$$(i = 1, 2, 3; j = 1, 2). \tag{15.4a}$$

$$\left(L_j \vartheta_3 + L_{j.}^{s} u_s + M u_j - \frac{\rho}{\mu} F_j \right)_C = \varphi_j^{(2)}(x^j, 0, t); \tag{15.4b}$$

$$\left(N_{3.}^{;s} \vartheta_s + \frac{\mu}{\lambda + 2\mu} M u_3 - \frac{\rho}{\lambda + 2\mu} F_3 \right)_C = \varphi_3^{(2)}(x^j, 0, t); \tag{15.4c}$$

$$(j, s = 1, 2).$$

The condition (15.4a), after application of the interpolation formulas, will lead to equations of geometric connectivity relative to the generalized coordinates $u_i(p, q)$ and $\vartheta_i(p, q)$. In fact, making use of eqs.(14.4a) - (14.4b), we obtain from eq.(15.4a): /279

* We recall that on the contour surface the coordinates x^j are connected by the equation of the contour of the basic surface.

$$\sum_{(p,q)} u_i(p, q)\,[\varphi_{pq}(x^j)]_C = \varphi_i(x^j, 0, t);$$

$$\sum_{(p,q)} \vartheta_i(p, q)\,[\psi_{pq}(x^j)]_C = \varphi_i^{(1)}(x^j, 0, t)$$

$$(i = 1, 2, 3;\ j = 1, 2). \tag{15.5}$$

Equations (15.5) must be set up for segments of arc of the contour C belonging to the base regions of approximation to which the sums $\sum_{(pq)}$ are extended. It can be similarly shown that the connectivity determined by eqs.(15.4b) - (15.4c) in the general case is not geometrical. We shall term it kinematic, although, in contrast to the kinematic connectivity of classical dynamics, its equations contain derivatives of the second order with respect to the time t of the generalized coordinates, if we confine ourselves to the three first terms of the expansions on the right-hand sides of eqs.(15.3a) to eqs.(15.3b).

2. Second Boundary Problem

This problem has been considered previously (III, Sect.13) in the formulation that best corresponds to the method under study.

Making use of the boundary conditions in the form of the equations of Chapter III (III, 13.9a), (III, 13.9b) we again find equations of connectivity analogous to those considered above. We note here that, retaining only the two first terms on the right-hand sides of the expansions (III, 13.6) we obtain, in the general case, one equation of geometric connectivity, and one equation of kinematic connectivity, which results directly from a consideration of the left-hand sides of eqs.(III, 13.9a), (III, 13.9b).

We will not write out the equations of these connectivities. Let us discuss only the cases in which the kinematic connectivity degenerates, as already discussed in our consideration on the basic boundary problems.

If the conditions of attachment of the contour of the basic surface include in themselves the conditions that some displacement component on the contour C shall vanish, then the wave operator M for this component will be transformed, as will be seen from (III, 6.3a), into a Laplace operator on the contour C. In this case the equations of kinematic connectivity resulting from the conditions (15.4b) - (15.4c) are transformed into equations of geometrical connectivity. Obviously, this does not apply to the derivatives of the operator M with respect to the coordinates x^i (i = 1, 2). These derivatives appear when introducing, into the expansions of $u_i^{(z)}$ in powers of z, terms containing z in degrees higher than the second. /280

In conclusion, we will make a brief statement on the correspondence between the boundary conditions and the equations of motion. Let us turn again to (III, Sect.13). There, we showed that the boundary conditions obtained as a result of the method of successive approximation cannot be satisfied with the same relative accuracy as the system of equations determining the displacements of the points of the basic surface.

These conclusions do not extend to the method studied here, since we do not exclude the components ϑ_j by means of successive approximations. The validity of the above becomes obvious from a comparison of the number of equations of motion and the number of equations of connectivity.

3. Initial Conditions

The initial conditions were considered previously (III, Sect.14), where we used the method of successive approximation, which, as already mentioned, will not be applied here.

Making use of the conditions (III, 14.1) and their expansions in series in ascending powers of z of the form (III, 14.2a), and also making use of eqs.(15.3a) - (15.3b), we find a system of initial conditions similar to the system considered above (III, Sect.14). Without writing out these conditions again, let us first pose the question whether the initial conditions correspond to the order of the system of differential equations of motion mentioned before (III, Sect.14). A definite answer can be given here only for the case of a linear dependence between the displacements $u_i^{(z)}$ and the coordinates z. This case corresponds to the representation of the strain tensor components by eqs.(14.1a) - (14.1b). Then the number of initial conditions containing the initial values $u_{io}(p, q)$, $\vartheta_{io}(p, q)$, $\dot{u}_{io}(p, q)$, $\dot{\vartheta}_{io}(p, q)$ $(i = 1, 2, 3)$ will be twelve for each pair of numbers (p, q). The system of equations of motion consisting of six differential equations of the second order for each pair of numbers (p, q) will be of an order equal to the number of initial conditions.

If, into the expansions of the displacement-vector and strain-tensor components, we introduce terms with the factor z^2, then, considering the cases of degeneration of the kinematic boundary conditions, we find that the number of initial conditions will increase to eighteen, while the system of differential equations of motion, as shown in the following Section, will consist of three equations of the sixth order and three of the second order, for each pair of numbers (p, q).

The initial conditions here will contain $u_{io}(p, q)$, $\dot{u}_{io}(p, q)$, $\ddot{u}_{io}(p, q)$, $\dddot{u}_{io}(p, q)$, $\vartheta_{io}(p, q)$ and $\dot{\vartheta}_{io}(p, q)$. There are not enough of these conditions, and we must consider the next terms of the expansions of the displacement vector components with the factor z^3. If, in this case, the relative accuracy of the equations of motion and the equations of connectivity remains unchanged, then the number of initial conditions increases to twenty-four for each pair of numbers (p, q); these will now include the initial values of the derivatives of the fourth and fifth order with respect to time, $u_{io}^{(IV)}$ (p, q) and $u_{io}^{(V)}$ (p, q). Thus, to solve the problem, the initial conditions must be satisfied with a higher degree of accuracy than the relative accuracy of the /281

equations of motion and the equations of connectivity.

Section 16. Equations of Motion of the Shell

If the boundary conditions lead to equations of geometric or degenerate kinematic connectivity, we can make use of the Ostrogradskiy-Hamilton principle to set up the equations of motion of the nodes of the interpolation net on the basic surface of the shell.

The Ostrogradskiy-Hamilton principle is expressed, as we know, by the following variational equation:

$$\int_{t_1}^{t_0} (\delta A + \delta L)\, dt = 0, \tag{16.1}$$

where δA is the elementary work of the nonconservative forces performed on passage of the points of the system from the trajectories of actual motion onto the trajectories of comparison, and L is the Lagrange function of the shell "as a whole".

If we confine ourselves to the first two terms on the right-hand sides of eqs.(15.3a) - (15.3b), and represent the strain tensor components by eqs.(14.1a) - (14.1b), then the equations of motion of the discrete-continuous system replacing the shell will be of the form of Lagrange equations of the second kind:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_i(p,\ q)} - \frac{\partial L}{\partial u_i(p,\ q)} = Q_i(p,\ q), \tag{16.2a}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}_i(p,\ q)} - \frac{\partial L}{\partial \vartheta_i(p,\ q)} = \Phi_i(p,\ q), \tag{16.2b}$$

$$(i = 1,\ 2,\ 3).$$

where Q_i and Φ_i are generalized nonconservative forces.

However, as we know, we have the right to include also conservative forces in these generalized forces, if this can help to simplify solution of the problem.

If the equations of kinematic connectivity degenerate on consideration /282 of three terms in the expansions of the displacement vector in powers of the coordinate z, then we can again make use of the Ostrogradskiy-Hamilton principle (16.1) in setting up the equations of motion, but in this case the Lagrange function L will contain the second and third time derivatives of the generalized coordinates $u_i(p,\ q)$ and, as before, the first time derivatives of the generalized coordinates $\vartheta_i(p,\ q)$. We can convince ourselves of this by considering the right-hand sides of eqs.(15.3a) - (15.3b). Thus, under the adopted assumptions on the equations of connectivity, the equations of motion of the discrete-continuous system replacing the shell will now be of the fol-

lowing form:

$$\frac{d^3}{dt^3}\frac{\partial L}{\partial \dddot{u}_i(p, q)}-\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{u}_i(p, q)}+\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_i(p, q)}-$$

$$-\frac{\partial L}{\partial u_i(p, q)}=Q_i(p, q), \qquad (16.3a)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}_i(p, q)}-\frac{\partial L}{\partial \vartheta_i(p, q)}=\Phi_i(p, q)$$

$$(i=1, 2, 3). \qquad (16.3b)$$

We call the reader's attention to the difference in the orders of the equations entering into the subsystems (16.3a) - (16.3b). This difference of orders, under the method of reduction here adopted, will occur when an odd number of terms is retained on the right-hand sides of the expansions in tensor series of the displacement vector components $u_i^{(z)}$ in ascending powers of z. With an even number of terms retained in the expansions, the order of the equations entering into the subsystems (16.3a) - (16.3b) will be the same*. This assertion is in particular illustrated by eqs.(16.2a) - (16.2b).

If a shell has a part of the contour surface free of connectivity, then the equations of kinematic connectivity, resulting from the conditions (15.4b) to (15.4c) will not degenerate into equations of geometric connectivity, and the Ostrogradskiy-Hamilton principle will not be applicable, at least not without additional investigation.

In these cases, we may give up the method of reduction based on consideration of six functions of u_i and ϑ_i of a point of the basic surface of the shell, permitting us to carry the reduction problem to completion; instead, we may use approximate representations of the displacement vector by polynomials arranged in powers of z, which were considered in Chapter III in our study of reduction methods relying on the general equation of dynamics. In this case, such difficulties will not arise, since the wave operator M appears in the /283 relations (15.3a) - (15.3b) as a result of application of the Lamé equations in order to eliminate, from the expansions in Taylor tensor series, the covariant derivatives of the component u_i of the second and higher orders with respect to $x^3 = z$. At the same time, it can be stated that application of the Lamé equations improves the accuracy of the approximations.

We will not develop a version of the discrete-continuous method that is not connected with the use of Lamé equations.

Section 17. Concluding Remarks

Chapter IV covered a group of questions connected with the fundamental

* This statement supplements our paper read at the Conference on Shell Theory at Kazan in 1960

problem of the shell theory, which reduces to the construction of a mechanical system approximately equivalent, according to some criterion, to the shell as a three-dimensional elastic body.

As the analytic criterion of approximate equivalence, we selected the magnitude of the quadratic deviation of some function characterizing the state of the mechanical system to be constructed from the corresponding function characterizing the state of the three-dimensional body, namely, the shell. As the function we chose the density of the Lagrange functions, the body forces, or the surface forces, depending on the specific problem involved. The solution of various physical problems was unified by the general requirement that these quantities for the approximately equivalent system show minimum deviation from the same quantities for the three-dimensional body, the shell.

Thus, Chapter IV contains the solution of a series of problems of approximation functions, which are reflected in the mechanics of shells. We have therefore also included in Chapter IV the first principles of the theory of the construction of a discrete-continuous system replacing the shell, and used interpolation formulas to find the required approximation.

In meaning, the construction of a discrete-continuous system replacing a shell is close to the finite-difference method.

The proposed method differs from the finite-difference method, however, in being more exact, since construction of the equations of motion is based on the operations of integration required for the calculation of both kinetic and potential energy. The operation of integration somewhat smoothes the errors introduced by the interpolation formulas. One of the major error sources still persists, namely, the approximation formulas of connectivity that result from the boundary conditions.

A shortcoming of the method is the complexity of the equations of motion and the boundary conditions. The field of applicability of the method therefore encompasses all problems where the use of methods of the Bubnov-Galerkin type involves fundamental difficulties in constructing the systems of approximation functions. /284

An elementary example of such problems is the problem of the vibrations of a rectangular plate with mixed conditions on each side of the rectangular contour of its middle surface.

An advantage of the method is the simplicity of programming in calculation on high-speed electronic computers.

CHAPTER V

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INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS OF THE THEORY OF SHELLS

Section 1. General Characteristics of the Contents of the Concluding Chapter

The last Chapter contains a discussion of part of our results in the methods of solving the boundary conditions of shell theory, relying on the integro-differential and integral equations of the statics and dynamics of shells, resulting from the theorem of work and reciprocity. These studies were begun by us in 1939-1940 and are still going on at present (Bibl.23b-j).

During the past five years the possibility of applying the apparatus of integro-differential and integral equations to the solution of boundary problems of the shell theory has attracted the attention of many workers. Besides the methods indicated above, they have used other methods, based particularly on the integral relations generalized in the Green formulas of the theory of the Newtonian potential function. Limited space prevent us from giving a detailed analysis of the various methods of reducing the boundary problems of the shell theory to equivalent systems of integro-differential and integral equations*. Many questions of the theory of this reduction, including the problem of equivalence, existence, and uniqueness of the solutions of the equations set up by us will not be exhaustively answered here. We intend to return to them in the second part of this book.

The last Sections of this Chapter will contain the integro-differential /286 and integral equations of the dynamics of shells, together with special applications of the generalized reciprocal theorem proved in Chapter II.

Section 2. Elementary Solutions of Three-Dimensional Problems of Elasticity Theory Containing Singular Points and Lines

In the first Section of this Chapter we applied the method of constructing integro-differential equations of the shell theory based on the introduction of solutions of the three-dimensional problem of elasticity theory containing singularities arranged along a certain segment of a straight line. We used this method earlier (Bibl.23b) in 1939 - 1940, and will take the results, given below, from that work. We shall consider solutions with singularities of the three-dimensional static problem of the theory of elasticity found for a linearly deformed medium.

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* During preparation of this work for the press, the book by D.V.Vaynberg and A.L.Sinyavskiy (Bibl.17) appeared which contains a brief discussion of the method of construction of integro-differential and integral equations of the shell theory given by us and other authors, together with several applications to the theory of specific boundary problems, including the problems of the equilibrium of cylindrical notched shells. It also gives (Bibl.17) a bibliography which is incomplete but still deserves attention.

It is well known that displacements in an unbounded elastic medium, due to the action of a single concentrated force directed along the axis OY_i of a rectangular Cartesian coordinate system y_i and applied to the point with coordinates η_i, are of the following form*:

$$\theta_{(i)\,i}(y_j,\ \eta_j) = \frac{1}{24\pi G}\left\{\frac{5-6\nu}{1-\nu}\,\frac{1}{r} + \frac{r^2}{2(1-\nu)}\,\frac{\partial^2}{\partial y_i^2}\left(\frac{1}{r}\right)\right\}; \tag{2.1a}$$

$$\theta_{(i)\,k}(y_j,\ \eta_j) = \frac{1}{24\pi G}\left\{\qquad\qquad + \frac{r^2}{2(1-\nu)}\,\frac{\partial^2}{\partial y_i \partial y_k}\left(\frac{1}{r}\right)\right\}$$

$$(i,\ j,\ k = 1,\ 2,\ 3),\ (i \neq k). \tag{2.1b}$$

where

$$G = \frac{E}{2(1+\nu)}; \quad r = \sqrt{(y_1-\eta_1)^2 + (y_2-\eta_2)^2 + (y_3-\eta_3)^2}. \tag{2.2}$$

If, instead of a single force, we apply at the point $M(\eta_j)$ the arbitrary forces Y_k directed along the coordinate axes, then the displacements corresponding to these forces will be expressed by the equalities:

$$\theta_i = \sum_{k=1}^{3} Y_k \theta_{(k)\,i} \qquad (i = 1,\ 2,\ 3). \tag{2.3}$$

The stresses corresponding to the displacements (2.1a) - (2.1b) have the /287 form:

$$\vartheta_{(i)\,ii} = -\frac{1}{8}\,\frac{1-2\nu}{\pi(1-\nu)}\left[\frac{y_i-\eta_i}{r^3} + \frac{3}{1-2\nu}\,\frac{(y_i-\eta_i)^3}{r^5}\right]; \tag{2.4a}$$

$$\vartheta_{(i)\,kk} = -\frac{1}{8}\,\frac{1-2\nu}{\pi(1-\nu)}\left[-\frac{y_i-\eta_i}{r^3} + \frac{3}{1-2\nu}\,\frac{(y_i-\eta_i)(y_k-\eta_k)^2}{r^5}\right]; \tag{2.4b}$$

$$\vartheta_{(i)\,ik} = -\frac{1}{8}\,\frac{1-2\nu}{\pi(1-\nu)}\left[\frac{y_k-\eta_k}{r^3} + \frac{3}{1-2\nu}\,\frac{(y_i-\eta_i)^2(y_k-\eta_k)}{r^5}\right]; \tag{2.4c}$$

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* Cf., for instance (Bibl.9b) or E.Trefftz, Mathematical Theory of Elasticity, ONTI, 1934, pp.39-40.

$$\vartheta_{(i)\,jk} = -\frac{1}{8}\frac{1-2\nu}{\pi(1-\nu)}\left[\qquad + \frac{3}{1-2\nu}\frac{(y_i-\eta_i)(y_j-\eta_j)(y_k-\eta_k)}{r^5}\right]$$

$$(i,\ j,\ k=1,\ 2,\ 3;\ i \neq j \neq k). \tag{2.4d}$$

The arbitrary system of forces Y_k applied at the point $M(\eta_j)$ creates a field of stresses defined by the equalities

$$\vartheta_{ij} = \sum_{k=1}^{3} Y_k \vartheta_{(k)\,ij} \qquad (i,\ j=1,\ 2,\ 3). \tag{2.5}$$

Let us now consider the expressions for the displacement vector components θ_i and the corresponding stress tensor components in a curvilinear system of coordinates by means of which the space inside the shell is arithmetized, under specification, as indicated below, of the direction of the single concentrated force.

Between the Cartesian coordinates y_i and the internal curvilinear coordinates x^i of the points of the shell there exist relationships expressed by the formulas of direct and inverse transformation:

$$x^i = x^i(y_j); \quad y_j = y_j(x^i) \qquad (i,\ j=1,\ 2,\ 3). \tag{2.6}$$

If we put $x^3 = 0$, then their relations

$$y_i = y_i(x^1,\ x^2,\ 0) \qquad (i=1,\ 2,\ 3) \tag{2.7}$$

will be the equations of the basic surface of the shell. Below, we will primarily consider shells of constant thickness. In this case, the basic surface will coincide with the middle surface, and the boundary surfaces of the shell will be included in the system of coordinate surfaces.

Let us now assume that at some point M of the shell a force is applied having the components

$$F^k_{(i)} = \delta^k_i, \tag{2.8}$$

directed along the tangent to the coordinate line x^i. Let us find its components in the rectangular Cartesian coordinate system y_j. Making use of the formulas for the transformation of the contravariant vector components (I, 5.5) and the equalities (2.6), we obtain /288

$$Y_{(i)\,k} = F^j_{(i)}\left(\frac{\partial y_k}{\partial x^j}\right)_M = \delta^j_i\left(\frac{\partial y_k}{\partial x^j}\right)_M = \left(\frac{\partial y_k}{\partial x^i}\right)_M \tag{2.9}$$

The index (i) of the force components in the rectangular system of Cartesian coordinates shows that they are force components directed along a tangent to the coordinate line x^i of the curvilinear coordinate system.

Equations (2.3), on choice of the force determined by eqs.(2.8) - (2.9), take the following form:

$$v_{(i)j} = \sum_{k=1}^{3} Y_{(i)k} \vartheta_{(k)j} = \sum_{k=1}^{3} \left(\frac{\partial y_k}{\partial x^i}\right)_M \vartheta_{(k)j} \quad (i = 1, 2, 3). \tag{2.10}$$

where $v_{(i)j}$ are the components of the vector of the displacements caused in the elastic medium by the concentrated force defined by eqs.(2.8) - (2.9). These components express the displacement vector in the rectangular Cartesian coordinate system OY_i.

Returning again to the coordinate system x^i, let us find the covariant components $u_{(i)j}$ of the vector of displacements caused by the action of the force $F^*_{(i)}$. We obtain

$$u_{(i)j} = \frac{\partial y_k}{\partial x^j} v_{(i)k} = \frac{\partial y_k}{\partial x^j} \left(\frac{\partial y_p}{\partial x^i}\right)_M \vartheta_{(p)k}. \tag{2.11}$$

A comparison of eqs.(2.11) with (I, 6.3) shows that the quantities $u_{(i)j}$ found by us possess peculiar tensor properties. These quantities may be considered as covariant components of the vector at the point $N(x^i)$. But being functions of the pair of points M and N, they are components of the covariant tensor of rank two, connected with these points*. Further, from eqs.(2.5) and (2.9) we find:

$$\vartheta_{(i)jk} = \sum_{p=1}^{3} Y_{(i)p} \vartheta_{(p)jk} = \sum_{p=1}^{3} \left(\frac{\partial y_p}{\partial x^i}\right)_M \vartheta_{(p)jk}. \tag{2.12}$$

Again applying the transformation formulas (I, 6.3), we obtain the contravariant components of the stress tensor in the curvilinear coordinate system /289

* We shall not dwell on the analogy between these quantities and the so-called "intermediate" tensor components. Cf.I.Schouten and D.Struik, Introduction to New Methods of Differential Geometry, ONTI, 1939, p.29.

connected with the shell.

$$\sigma_{(i)}^{\cdot jk} = \frac{\partial x^j}{\partial y^r} \frac{\partial x^k}{\partial y^s} \left(\frac{\partial y_p}{\partial x^i} \right)_M \vartheta_{(p)}^{\cdot rs}$$

$$(i, j, k, p, r, s = 1, 2, 3). \tag{2.13}$$

We recall that in a rectangular Cartesian coordinate system the noninvariant equality

$$\vartheta_{(p)}^{\cdot rs} \doteq \vartheta_{(p)rs}. \tag{a}$$

is satisfied.

Equations (2.13) show that the quantities $\sigma_{(i)}^{jk}$ are components of a third-rank tensor of two points. At point M, these quantities are components of the vector with the subscripts (i) and (p). At point N, however, they are components of a second-rank tensor.

Let us pass now to the construction of new solutions of the three-dimensional problem of the theory of elasticity with singularities. The solutions we have considered for the homogeneous static equations of the elasticity theory satisfy these equations for all values of the coordinates, except for the coordinates of the point of application of the concentrated force. This point is singular. In it, the displacements become infinite of the order r^{-1} as $r \to 0$, and the stresses become infinite of the order r^{-2}. Such singularities are encountered in the Newtonian potential function. On the basis of the derived solutions we can find a series of new solutions of the homogeneous static equations of the theory of elasticity with a continuous distribution of the singular points along a certain line.

Let us assume that this line is a segment of a straight line, of length 2Mh, where M > 1. Let the coordinates of the middle of this segment in the rectangular Cartesian coordinate system be ζ_i (i = 1, 2, 3). Then, the coordinates of the points of the segment can be expressed by the equalities:

$$\eta_i = \zeta_i + \alpha a_i \qquad (i = 1, 2, 3), \tag{2.14}$$

where

$$\sum_{i=1}^{3} a_i^2 = 1. \tag{b}$$

Here, a_i are the direction cosines of the segment, and α is the distance from the middle of the segment with fixed positive direction.

Let us distribute on this segment the force load of linear density $q(\alpha)$, directed along one of the axes of the local coordinate basis. Assume that in the rectangular Cartesian coordinate system y_i, the direction of the forces /290 $q(\alpha)d(\alpha)$ coincides with the direction of the axis OY_j. Then, the displacements caused by these forces will be expressed by the equalities

$$w_{(j)k} = \int_{-Mh}^{+Mh} q(\alpha)\,\theta_{(j)k}(\alpha)\,d\alpha \qquad (j,\ k = 1,\ 2,\ 3). \tag{2.15}$$

where the functions $\theta_{(j)k}(\alpha)$ are determined by the relations (2.1a) - (2.1b) if the coordinates of the point of application of the forces are expressed by the equalities (2.14).

The functions $w_{(j)k}$ satisfy the homogeneous static equations of the elasticity theory for all values of the coordinates, except the coordinates of the points on the segment bearing the force load. These coordinates are expressed by eqs.(2.14). This segment is thus a singular line for the functions $w_{(j)k}$. The load density can be selected under very wide assumptions relative to the analytic properties of the functions $q(\alpha)$.

Consider the functions $q(\alpha)$ causing the local perturbations. In other words, let us select a function $q(\alpha)$ such that the displacements and stresses due to the respective load will rapidly attenuate with increasing distance of the point $N(y_i)$ from the segment on which the load is distributed. We shall call such a load a focusing load, since it will subsequently permit us to separate part of the wanted field of displacement in the neighborhood of the singular line, and to "liquidate" the residual . For the construction of the focusing load we employ the method given elsewhere (Bibl.23b). Of course this method cannot be considered optimum, but we will not further discuss the methods of optimum choice of the focusing load.

First let us analyze the conditions whose satisfaction enables us to represent the functions $\theta_{(j)}$, (α) in the form of series in ascending positive powers of α.

It will be clear from eqs.(2.1a) - (2.1b) that the singularity contained in the functions $\theta_{(j)k}$ (j, k = 1, 2, 3) depends on a factor of the form r^{-n}. The question of the possibility of expanding these functions in series in ascending powers of α therefore reduces to an analysis of the possibility of such expansion for the function r^{-n}. From eq.(2.2) we find

$$r^2(\alpha) = (y_1 - \zeta_1 - \alpha a_1)^2 + (y_2 - \zeta_2 - \alpha a_2)^2 + (y_3 - \zeta_3 - \alpha a_3)^2, \tag{c}$$

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* This term is borrowed from the book by C.Lanczos "Practical Methods of Applied Analysis", Fizmatgiz, 1961, p.220. The meaning of this term is extended by us.

or, in other words,

$$r^2(\alpha) = r_0^2 - 2\alpha r_0 \cos\varphi + \alpha^2, \tag{d}$$

where /291

$$r_0^2 = (y_1 - \zeta_1)^2 + (y_2 - \zeta_2)^2 + (y_3 - \zeta_3)^2 \tag{e}$$

and

$$2\alpha r_0 \cos\varphi = 2r_0 \cdot \alpha \tag{f}$$

is the doubled scalar product of the vectors $\vec{r}_0$ and $\vec{\alpha}$.

From eq.(d), we find

$$r^{-n}(\alpha) = r_0^{-n}\left(1 - \frac{2\alpha\cos\varphi}{r_0} + \frac{\alpha^2}{r_0^2}\right)^{-\frac{n}{2}} \tag{2.16a}$$

The expansion of $r^{-n}(\alpha)$ in a series in ascending powers of α is possible when the inequality

$$\left| -\frac{2\alpha\cos\varphi}{r_0} + \frac{\alpha^2}{r_0^2} \right| < 1, \tag{2.16b}$$

is satisfied, or, strengthening the inequality,

$$\frac{2|\alpha|}{r_0} + \frac{\alpha^2}{r_0^2} < 1. \tag{g}$$

Hence, we find*

$$r_0 > \frac{|\alpha|}{\sqrt{2} - 1} \simeq 2{,}5|\alpha|. \tag{2.17}$$

Here, of course, we have taken a positive value for $\sqrt{2}$. The maximum value of $|\alpha|$ is Mh. Consequently, at the points satisfying the condition

$$r_0 > 2{,}5\,Mh, \tag{2.18}$$

* The estimate [eq.(2.17)] is too high; cf. Sect.3. See also M.A.Lavrent'yev and B.V.Shabat, Methods of the Theory of Functions of a Complex Variable, Gostekhizdat, 1951, pp.501 - 502.

the functions $\theta_{(j)k}(\alpha)$ can be represented by the convergent series in ascending powers of α:

$$\theta_{(j)k}(\alpha)=\sum_{n=0}^{\infty}\theta_{(j)k}^{(n)}\alpha^{n}\qquad (j,\ k=1,\ 2,\ 3). \tag{2.19}$$

These series will absolutely converge for all values of α lying on the segment (-MH, +Mh)*. On the basis of eqs.(2.19), the functions $w_{(j)k}$ defined by eqs.(2.15) can be represented by expansions of the form /292

$$w_{(j)k}=\sum_{n=0}^{\infty}m_{n}\theta_{(j)k}^{(n)}, \tag{2.20}$$

where the coefficients m_n are defined by the formulas

$$m_{n}=\int_{-Mh}^{+Mh}q(\alpha)\,\alpha^{n}d\alpha. \tag{2.21}$$

Let us assume at first that the functions $q(\alpha)$ are everywhere bounded on the interval (-Mh, +Mh) and are continuous, except for a finite number of points of discontinuity of the first kind. Assume further that on the continuity intervals the function $q(\alpha)$ is represented by polynomials of degree N, where N is for the time being an arbitrary number. Then, coefficients of these polynomials can always be selected such that the equalities

$$m_{n}=0\ \ (n=0,\ 1,\ 2,\dots,\ N-1) \tag{2.22}$$

shall be satisfied.

Equations (2.22) form a system of linear algebraic equation from which the N coefficients of the polynomial representing the function $q(\alpha)$ can be determined in terms of the $(N+1)^{th}$ coefficient if the function $q(\alpha)$ is continuous over the entire interval (-Mh, +Mh). We shall consider the case of the discontinuous function $q(\alpha)$ somewhat later.

It follows from eq.(2.21) that the coefficient m_N is of a relative order not lower than the order of the quantity $(Mh)^{N+1}$. Consequently, the displacements $w_{(j)k}$ defined by eqs.(2.20) will be of an order not lower than the order of the ratio $(Mh)^{N+1}:r_0^{N+1}$ over the entire region in which r_0 satisfies the in-

* On absolute and uniform convergence of these expansions, cf., for instance, E.T.Whittaker and G.N.Watson, Course in Modern Analysis, Vol.II, Gostekhizdat, 1934, pp.91-92.

equality (2.17).

Since, in this region, the inequality

$$\frac{Mh}{r_0} < \frac{1}{2{,}5}, \tag{i}$$

is satisfied, it follows from the above that in this region we can construct displacements $w_{(i)k}$ which are negligibly small in absolute magnitude. Let us investigate this question in more detail, considering the concrete construction of a function $q(\alpha)$ with the above-noted focusing properties.

Let us consider the density of the load $q(\alpha)$ determined over the interval $(-Mh, +Mh)$ as follows:

a) the function $q(\alpha)$ is piecewise-continuous over the interval $(-Mh, +Mh)$;

b) the function $q(\alpha)$ is zero over the interval $(-\epsilon h, +\epsilon h)$; $(h - \epsilon h, h + \epsilon h)$, $(-h - \epsilon h, -h + \epsilon h)$, where $\epsilon \ll 1$;

c) the function $q(\alpha)$ is normed by the condition /293

$$\int_{-h}^{+h} q(\alpha)\, d\alpha = 1. \tag{2.23}$$

Let us denote the value of the piecewise-continuous function $q(\alpha)$ over the interval $(h + \epsilon h, Mh)$ by $q_1(\alpha)$; over the interval $(\epsilon h, h - \epsilon h)$ by $q_2(\alpha)$; over the interval $(-h + \epsilon h, -\epsilon h)$ by $q_3(\alpha)$; over the interval $(-h - \epsilon h, -Mh)$, by $q_4(\alpha)$.

Let us now impose on the functions $q_1(\alpha), \ldots, q_4(\alpha)$ the condition that the load on the intervals $(0, Mh)$ and $(-Mh, 0)$ be self-balanced:

$$\int_0^{Mh} \alpha^n q(\alpha)\, d\alpha = 0; \quad \int_0^{-Mh} \alpha^n q(\alpha)\, d\alpha = 0 \tag{2.24}$$
$$(n = 0, 1, 2, \ldots, N).$$

Consider, for example, the first group of conditions (2.24). From these conditions, it follows that

$$\int_{h+\varepsilon h}^{Mh} p_n(t)\, q_1(t)\, dt = -\int_{\varepsilon h}^{h-\varepsilon h} p_n(\alpha)\, q_2(\alpha)\, d\alpha, \tag{2.25}$$

where $p_n(t)$ and $p_n(\alpha)$ are certain polynomials.

Assume that the function $q_2(\alpha)$ is assigned. For definiteness and certain simplifications in the subsequent calculations, let us put

$$M = 2 - \varepsilon. \tag{k}$$

Let us perform the substitution of the variables bringing the integration intervals in eq.(2.25) to the standard interval (-1, +1). Let us put

$$t = \frac{1}{2} h (1 - 2\varepsilon) \left(z + \frac{3}{1 - 2\varepsilon} \right); \tag{l}$$

$$\alpha = \frac{1}{2} h (1 - 2\varepsilon) \left(z + \frac{1}{1 - 2\varepsilon} \right) = \frac{1}{2} h (1 - 2\varepsilon) \left(z' + \frac{3}{1 - 2\varepsilon} \right). \tag{m}$$

Let us select the polynomial $p_n(t)$ such that, on this substitution of the variable t, it shall be transformed into the Legendre polynomial $P_n(z)$:

$$p_n(t) = P_n(z). \tag{n}$$

Then the polynomial $p_n(\alpha)$ is transformed as follows:

$$p_n(\alpha) = P_n(z') = P_n\left(z - \frac{2}{1 - 2\varepsilon} \right). \tag{o}$$

Let us put further /294

$$q_1(t) = Q_1(z); \; q_2(\alpha) = Q_2(z). \tag{2.26}$$

Then eq.(2.25) takes the following form:

$$\int_{-1}^{+1} P_n(z)\, Q_1(z)\, dz = -\int_{-1}^{+1} P_n\left(z - \frac{2}{1 - 2\varepsilon} \right) Q_2(z)\, dz. \tag{2.27}$$

Equation (2.27) determines the coefficients of the expansion of the function $Q_1(z)$ in Legendre polynomials, if, as already specified, we prescribe the function $Q_2(\alpha)$.

Assume that the function $Q_1(z)$ can be approximately represented by the polynomial $Q_1^{(N)}(z)$ of degree N. Then this polynomial will be represented by an expansion in Legendre polynomials

$$Q_1^{(N)}(z) \sim \sum_{n=0}^{N} B_n P_n(z), \tag{2.28}$$

where

$$B_n = -\frac{2n+1}{2} \int_{-1}^{+1} P_n\left(z - \frac{2}{1-2\varepsilon}\right) Q_2(z)\, dz. \tag{2.29}$$

By increasing N in formula (2.28), we obtain an infinite sequence of functions $Q_1^{(N)}(z)$.

Speaking generally, this sequence can be divergent, but it always retains a certain mechanical meaning. Now let us consider the load $q_1^{(N)}(t)$. Let us construct the function

$$P_1^{(N)}(t) = \frac{h(1-2\varepsilon)}{N} q_1^{(N)}(t).$$

This function may be regarded as the mean value of the resultant of forces with a distribution density $q_1^{(N)}(t)$ applied to 1/N part of the interval over which these forces are distributed.

If there exists $\lim_{N\to\infty} P_1^{(N)}(t) = P(t)$ not equal to zero, then we may assert that the singularities corresponding to the limiting values of the density $q_1^{(N)}(t)$ are the result of the continuous distribution of concentrated forces of finite magnitude over the interval $(h + \epsilon h,\ 2h - \epsilon h)$. If $\lim_{N\to\infty} P_1^{(N)}(t)$ does not exist, on this interval there are distributed singularities of the force field of the type of force dipoles, etc.. Thus the limit of the sequence /295
$Q_1^{(N)}(z)$ determines the singularities of the force field constructed by us beyond the limits of the interval $(-h, +h)$.

The function $q_4^{(N)}(t)$ is constructed similarly to the function $q_1^{(N)}(t)$ if the function $q_3(\alpha)$ is prescribed. This exhausts the question of construction of the function $q(\alpha)$ over the interval $(-Mh, +Mh)$, Of course, this extension of the class of functions $q(\alpha)$ demands a corresponding extension of the integrability conditions of these functions. We will not further discuss this ques-

tion and will assume that the necessary extension of the integrability conditions can be found.

Thus, by selecting the functions $q_3(\alpha)$ and $q_3(\alpha)$ such that the condition (2.23) is satisfied, we can construct the functions $q_1(\alpha)$ and $q_4(\alpha)$, starting out from the conditions (2.24). But then, according to eqs.(2.20) - (2.21), the displacements $w_{(i)k}$ will vanish beyond the sphere S of radius $r_0 > 2.5$ Mh. In this region external to the sphere S, the stresses corresponding to the displacements $w_{(i)k}$ and the derivatives of those quantities will also vanish. Within the sphere S the displacements found by us will satisfy the continuity conditions and the equations of elasticity theory everywhere, except on the line on which there are singularities.

The density $q(\alpha)$ with the considered properties focuses the field of displacements and stress near the segment on which it is distributed.

If neither $\lim_{N\to\infty} Q_1^{(N)}(z)$ nor $\lim_{N\to\infty} Q_4^{(N)}(z)$ exist, then eqs.(2.22) will be satisfied only for limited values of N. In this case, the field of displacements and stresses will not disappear beyond the limits of the sphere S, i.e., the focusing properties of the load $q(\alpha)$ will be weakened. This can apparently take place everywhere on condition that $|q(\alpha)|$ is bounded on the interval (-Mh, +Mh).

In conclusion, let us make an approximate evaluation of the variability of the displacements $w_{(i)k}$ and of the corresponding stresses in the neighborhood of a segment of the straight line over which a load of density $q(\alpha)$ is distributed. Let us return to eqs.(2.1a) - (2.1b). These formulas can be represented in the following form:

$$\theta_{(i)\,i}=\frac{A}{r}\,[C+B\cos^2(ry_i)];\qquad \theta_{(i)\,k}=\frac{AB}{r}\cos(ry_i)\cos(ry_k), \tag{2.30}$$

where

$$A=\frac{1}{24\pi G};\quad B=\frac{3}{2(1-\nu)};\quad C=\frac{9-12\nu}{2(1-\nu)}.$$

It follows that the displacement vector components determined by eqs.(2.30) /296 can be represented as

$$\theta_{(i)\,k}=\frac{\Phi_{(i)\,k}(\alpha)}{}, \tag{2.31}$$

where $\Phi(\alpha)$ is a certain bounded function of the parameter α.

Of course, this function also depends on the coordinates of the point $N(y_i)$ at which the displacements are determined and on the coordinates ζ_i of the cen-

ter of the segment on which the load is placed.

Further, applying the theorem of the integral mean, we find

$$\int_a^b q(\alpha)\,\theta_{(i)k}\,d\alpha = \Phi_{(i)k}(\beta)\,q(\beta)\int_a^b \frac{d\alpha}{r}. \tag{p}$$

where β lies in the interval (a, b).

If the point $N(y_1)$ at which the displacements are determined lies outside the lines bearing the load, then the integral $\int_a^b \frac{da}{r}$ will be nonsingular.

Integrating, we find from eq.(p),

$$\int_a^b q(\alpha)\,\theta_{(j)k}(\alpha)\,d\alpha = \Phi_{(j)k}(\beta)\,q(\beta)\ln\frac{b+p+r(b)}{a+p+r(a)}, \tag{2.32}$$

where $p = -r_0 \cos\varphi$ and r(a) and r(b) are the distances from the respective points a and b to the point $N(y_1)$.

If φ is zero or π, then the point N, at which the displacements $w_{(i)k}$ are determined, lies on the straight line bearing the load. Two cases must be distinguished here. If the point N does not lie on the part of the straight line over which the load is distributed, then the interval under consideration, as above, will not be singular.

In the case where the point N does lie on the interval (a, b), this integral will be improper, but with an existing Cauchy principal value. If the point N coincides with one end of the interval (a, b), then the integral $\int_a^b \frac{da}{r}$ will be divergent. As will be seen from eqs.(2.32) and (2.15), at these points the functions $w_{(j)k}$ have logarithmic features.

Of course all these conclusions are valid only for narrow classes of functions $q(\alpha)$, which in particular, admit of the application of the theorem of the integral mean. However we can always select functions $q_2(\alpha)$ and $q_3(\alpha)$ such that the conditions of applicability of the theorems of classical analysis will be satisfied. As for the functions $q_1(\alpha)$ and $q_4(\alpha)$, they lie on parts of the /297 straight line running outside the region filled by the material of the shell, which permits us to arrange them more arbitrarily than the functions $q_2(\alpha)$ and $q_3(\alpha)$.

Consider now the variability of the stress tensor components corresponding to the displacements $w_{(i)k}$ in the neighborhood of a singular line. Equations (2.4a) - (2.4d) can be put into the following form:

$$\vartheta_{(i)\,ii} = \frac{A_1}{r^2} \cos(ry_i)\,[1 + B\cos^2(ry_i)], \tag{2.33a}$$

$$\vartheta_{(i)\,kk} = \frac{A_1}{r^2} \cos(ry_i)\,[-1 + B\cos^2(ry_k)], \tag{2.33b}$$

$$\vartheta_{(i)\,ik} = \frac{A_1}{r^2} \cos(ry_k)\,[1 + B\cos^2(ry_i)], \tag{2.33c}$$

$$\vartheta_{(i)\,jk} = \frac{A_1 B}{r^2} \cos(ry_i)\cos(ry_j)\cos(ry_k). \tag{2.33d}$$

where

$$A_1 = -\frac{1 - 2\nu}{8\pi(1 - \nu)}.$$

It will be seen from eqs.(22.3a) - (2.33d) that the components of the stress tensor $\vartheta_{(i)rs}$ can be described by the following formula:

$$\vartheta_{(i)\,rs}(\alpha) = \frac{\Psi_{(i)\,rs}(\alpha)}{r^2}, \tag{2.34}$$

where $\psi_{(i)rs}$ is a function of the parameter α, and, of course, of the coordinates of the point $N(y_i)$ of the stress field $\vartheta_{(i)rs}$ and of the parameters determining the position of the load-carrying segments.

Applying again to the theorem of the integral mean, let us consider the integral

$$\int_a^b q(\alpha)\,\vartheta_{(i)\,rs}(\alpha)\,d\alpha = \Psi_{(i)\,rs}(\beta)\,q(\beta)\int_a^b \frac{d\alpha}{r^2}. \tag{q}$$

The meaning of the notation here used will be clear from the above exposition.

The integral $\int_a^b \frac{d\alpha}{r^2}$ is nonsingular if the point $N(y_i)$ lies outside the

straight line bearing the load $q(\alpha)$ or lies on a part of the straight line free from load. If the point $N(y_1)$ lies on a segment of the straight line over which the load is distributed with a density $q(\alpha)$, then this integral will be improper and divergent.

Assuming at first that the point $N(y_1)$ lies outside the segment of the straight line on which the function $q(\alpha)$ differs from zero, we find from eq.(q) that /298

$$\int_a^b q(\alpha)\,\vartheta_{(i)\,rs}(\alpha)\,d\alpha = \frac{\Psi_{(i)\,rs}(\beta)\,q(\beta)}{r_0\sin\varphi}\times$$

$$\times\left[\tan^{-1}\frac{b-r_0\cos\varphi}{r_0\sin\varphi}-\tan^{-1}\frac{a-r_0\cos\varphi}{r_0\sin\varphi}\right]. \qquad (2.35)$$

This equality confirms the above assertions on the properties of the integral under consideration, if we investigate the limit passage of the point $N(y_1)$ on the straight line bearing the load. In particular, when the point $N(y_1)$ approaches the segment with the nonzero load density $q(\alpha)$, the integral (q) increases without limit, but not more rapidly than the function r^{-1} as $r \to 0$.

All the above establishes the properties of variability of the stress field corresponding to the displacements $w_{(1)k}$ determined by eqs.(2.15) in the neighborhood of the singular line bearing the load of density $q(\alpha)$. We emphasize again that the analysis given here by no means exhausts all properties of the displacement fields $w_{(1)k}$ and of the corresponding stress fields, since it has been performed under simplified ideas as to the analytic properties of the function $q(\alpha)$.

Section 3. Integrodifferential and Integral Equations of the Statics of Shells, with Focusing Kernels

We give below a method of solving the boundary problems of statics of shells, relying on the theorem of work and reciprocity (II, Sect.12). We will confine the discussion for the time being to the formulation of this theorem, known from the linear theory of elasticity. The entire method can be considered as a development of the well-known method of Somigliano*.

It is well known that the theorem of work and reciprocity, or the Reciprocal Theorem, makes it possible to establish an interrelation between two fields of displacements and stresses induced in an elastic body by two systems of forces applied to it. We shall distinguish the main and auxiliary fields of displacements, stresses, and forces.

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* A.Love, Mathematical Theory of Elasticity, ONTI, 1935.

We will apply the term "basic" to the fields of displacements and stresses due to a load acting on the shell in accordance with the conditions of the boundary problem to be considered. These fields are usually unknown and must be determined. The notation of the components of the vector of basic displacements and the components of the tensor of basic stresses are known from the earlier Chapters of this book.

Let us pass to the consideration of the auxiliary displacement and stress fields and to the corresponding surface and body forces. Let us again consider an unbounded elastic medium and imagine that part of this medium is the shell we are studying. Let the unbounded medium be under the action of forces applied to a certain segment (-Mh, +Mh) of a straight line in the manner indicated in the preceding Section. /299

Let us direct the segment (-Mh, +Mh) of the load-carrying straight line along the normal to the middle surface of the shell. Let us assume that a point lying on the middle surface corresponds to the zero value of the parameter α on the load-carrying segment.

Let us further assume that the distribution density of the loads $q(\alpha)$ is determined by conditions (a) and (b) of the preceding Section, and also that it satisfies eqs.(2.23) - (2.24). Under these conditions, two segments of the singular line on which the function $q(\alpha)$ does not vanish will lie inside the shell. We note that under conditions (a) and (b) of the preceding Section the load-carrying segments do not intersect the middle surface nor the boundary surfaces of the shell. These surfaces are free from singular points both of the displacement field and of the stress field caused by the load distributed on the singular line.

Let us assume that the forces applied to the singular line are directed along the vector $\vec{e}_i$ of the local coordinate bases, where the index i is fixed. We recall that, under the assumptions adopted by us, the vector $\vec{e}_3$ directed along the normal to the undeformed middle surface has the same direction as the segment of the straight line bearing the load.

We shall apply the term "auxiliary" to those fields of displacements and stresses created both in the shell, and in a part of the unbounded elastic medium, by the load of the above-mentioned singular line. For these fields to exist in a shell cut out of the unbounded elastic medium, a system of surface forces, determined from the known stresses by the formulas (II, 8.2a - 8.2b), must be applied to the shell.

The surface forces so found, together with the forces distributed on that part of the singular line lying within the shell, form the system of auxiliary loads.

Let us consider the analytic expressions for the auxiliary displacements in the curvilinear system of coordinates x^i connected with the shell.

To avoid difficulties in determining the field of auxiliary displacements in the curvilinear coordinate system, let us first use a rectangular Cartesian

coordinate system, and then apply the formulas of transformation of the components of tensor quantities.

We shall retain the earlier notation for the load density $q(\alpha)$. The vector of the corresponding force is directed along the tangents to the coordinate line $\vec{e}_i$ of the curvilinear coordinate system. Let us find the components of /300 the vector density of the load $q^{(j)}(\alpha)$ in this system:

$$q^j(\alpha) = q(\alpha)\,\delta_i^j \qquad (i,\ j = 1,\ 2,\ 3). \tag{3.1}$$

Passing to rectangular Cartesian coordinates, we obtain a relation analogous to eq.(2.9) for the vector density components:

$$p_{(i)\,k}(\alpha) = q(\alpha)\left(\frac{\partial y_k}{\partial x^i}\right)_{M(\alpha)} \tag{3.2}$$

The field of displacements due to the load with the vector density $p_{(i)\,k} \times \times(\alpha)$ in the rectangular Cartesian coordinates will be determined by formulas analogous to eqs.(2.10) and (2.15):

$$)_j = \int\limits_{-Mh}^{+Mh} \sum_{k=1}^{3} q(\alpha)\left(\frac{\partial y_k}{\partial x^i}\right)_{M(\alpha)} \theta_{(k)\,j}(\alpha)\,d\alpha \qquad (i,\ j = 1,\ 2,\ 3). \tag{3.3}$$

Let us return now to the curvilinear coordinates. The covariant components of the vector of auxiliary displacements are expressed by an equality analogous to the relation (2.11):

$$v_{(i)\,j} = \frac{\partial y_k}{\partial x^j}\,w_{(i)\,k} = \frac{\partial y_k}{\partial x^j}\int\limits_{-Mh}^{+Mh} q(\alpha)\left(\frac{\partial y_p}{\partial x^i}\right)_{M(\alpha)} \theta_{(p)k}(\alpha)\,d\alpha \qquad (i,\ j,\ k,\ p = 1,\ 2,\ 3). \tag{3.4}$$

The signs of summation over k and p are omitted.

Equation (3.4) can be simplified by using a moving Cartesian system of coordinates and assuming that the axis OY_3 coincides with the straight line bearing the load, directed, as already stated, along the normal to the undeformed middle surface, i.e., along the coordinate line x^3 of the curvilinear coordinate system. In this case,

$$\left(\frac{\partial y_p}{\partial x^i}\right)_{M(\alpha)} = \delta_i^p (\sqrt{g_{pp}})_{M(\alpha)} \tag{a}$$

and eq.(3.4) takes the following form:

$$v_{(i)j} \doteq \frac{\partial y_k}{\partial x^j} \int_{-Mh}^{+Mh} (\sqrt{g_{ii}})_{M(\alpha)} q(\alpha)\, \theta_{(i)k}(\alpha)\, d\alpha. \tag{3.5}$$

This expression of auxiliary displacements is not invariant; the displacements $w_{(j)k}$ are defined by eqs.(2.15). The expressions of the stress tensor components and of the components of the surface forces defined according to the components $v_{(i)j}$ from Hooke's law will not be given here. /301

Applying the Reciprocal Theorem to the basic and auxiliary systems of displacements and loads, we find

$$\int_{-h}^{+h} q(\alpha)\, u_i(\alpha)\, d\alpha + \iint_{(S)} S^j_{(i)} u_j dS = \iint_{(S)} X^j v_{(i)j}\, dS + \iiint_{(V)} F^j v_{(i)j}\, dV. \tag{3.6}$$

where the $S^j_{(i)}$ denote the components of the auxiliary surface forces, X^j and F^j are the components of the surface and body forces of the basic system, while u_i are the covariant components of the vector of the principal displacement. The integrals $\iint_{(S)}$ extend over the boundary and contour surfaces of the shell and the integrals $\iiint_{(V)}$ extend over its volume.

The expression

$$\int_{-h}^{+h} q(\alpha)\, u_i(\alpha)\, d\alpha = \bar{u}_i(x^j) \qquad (j = 1,\ 2) \tag{3.7}$$

can be regarded the average value, with the weight $q(\alpha)$, of the displacement vector components u_i. This quantity is a function of the coordinates of the point $M(x^j)$ intersected by the straight line bearing the auxiliary load, and the middle surface of the shell.

In connection with eq.(2.23) the quantities $\bar{u}_i$ can be conventionally regarded as components of the displacements of the two-dimensional continuum studied in the shell theory.

The function $q(\alpha)$ can always be taken such that the condition

$$|u_i^{(0)}(x^i) - \bar{u}_i(x^i)| \leq Ah^m \qquad (3.8)$$

will be satisfied, where $u_i^{(0)}(x^i)$ are the displacements of the point $M_0(x^i)$ lying on the middle surface, and A is a constant.

The condition (3.8) assures the possibility of an approximate substitution of the integral (3.7), i.e., of the averaged displacements $\bar{u}_i$, by the displacements $u_i^{(0)}$ of the middle surface of the shell. This solves part of the general problem of reduction of the three-dimensional problem of the theory of elasticity to the two-dimensional problem of the theory of shells.

We shall now indicate an elementary method of constructing the function $q(\alpha)$ permitting us to satisfy condition (3.8). Assume that the displacement $u_i(\alpha)$ can be approximated by the polyonomial:

$$u_i(\alpha) \cong u_i^{(0)} + \alpha u_i^{(1)} + \alpha^2 u_i^{(2)} + \ldots + \alpha^N u_i^{(N)}. \qquad (3.9)$$

Then, the averaged displacements $\bar{u}_i(\alpha)$ can be represented in the form: /302

$$\bar{u}_i \cong \int_{-h}^{+h} q(\alpha)[u_i^{(0)} + \alpha u_i^{(1)} \ldots + \alpha^N u_i^{(N)}]\, d\alpha = \sum_{j=1}^{N} n_j u_i^{(j)}, \qquad (3.10)$$

where

$$n_j = \int_{-h}^{h} \alpha^j q(\alpha)\, d\alpha, \qquad (3.11)$$

α^j are the positive powers of the parameter α (but not of the components of the contravariant vector!).

On the basis of the properties of the function $q(\alpha)$, considered in Sect.2, let us select the functions $q_2(\alpha)$ and $q_3(\alpha)$ in the form of polynomials such that the conditions

$$n_j = 0 \;\; (j = 1, 2, \ldots, N). \qquad (3.12)$$

shall be satisfied.

If we confine ourselves to the relative accuracy adopted in the classical

theory and in Chapter III of this book in setting up the system of differential equations of motion of an element of the shell, then it is sufficient to put

$$n_1 = n_2 = 0. \tag{3.13}$$

Let us construct the polynomials $q_2(\alpha)$ and $q_3(\alpha)$ such that the functions $q(\alpha)$ shall be even over the interval $(-h, +h)$. In this case, even under condition (3.13) in the right-hand side of eq.(3.10), we shall have nonvanishing coefficients for the terms containing h^5 and higher orders of h, and the coefficients of $u_i^{(0)}$ will also be nonvanishing. It is not hard to see that for this it is sufficient to put

$$q_2(\alpha) = a_0 + \frac{a_1}{h}\alpha; \quad q_3(\alpha) = a_0 - \frac{a_1}{h}\alpha. \tag{3.14}$$

For definiteness we shall assume that the parameter ϵ in the conditions(a) and (b) imposed on the function $q(\alpha)$ in Sect.2 is 0.25. Then, we obtain

$$n_0 = \int\limits_{-0,75h}^{-0.25h}\left(a_0 - \frac{a_1}{h}\alpha\right)d\alpha + \int\limits_{0,25h}^{0,75h}\left(a_0 + \frac{a_1}{h}\alpha\right)d\alpha, \tag{b}$$

$$n_2 = \int\limits_{-0,75h}^{-0.25h}\left(a_0 - \frac{a_1}{h}\alpha\right)\alpha^2 d\alpha + \int\limits_{0,25h}^{0,75h}\left(a_0 + \frac{a_1}{h}\alpha\right)\alpha^2 d\alpha. \tag{c}$$

All the coefficients of the n_j with odd indices$_j$ vanish identically. Equating the coefficient n_0, according to the condition (2.23), to unity, and the /303 coefficient n_2 to zero, we find the coefficients a_0 and a_1. Equation (3.14) takes the following form:

$$q_2(\alpha) = \frac{15}{2h}\left(1 - \frac{26}{15}\frac{\alpha}{h}\right); \quad q_3(\alpha) = \frac{15}{2h}\left(1 + \frac{26}{15}\frac{\alpha}{h}\right). \tag{3.15}$$

We find the functions $q_1(\alpha)$ and $q_4(\alpha)$ from conditions (2.24), confining ourselves to their approximate representation resulting from eqs.(2.28) - (2.29).

We also note that this elementary method of constructing the functions $q(\alpha)$ involves analytic restrictions imposed on the components u_i of the displacement vector. These restrictions are expressed by the assumption that an approximate

representation of these components by polynomials according to eq.(3.9) is possible. Of course, by employing more general methods of constructing the functions $q(\alpha)$ than those just described, we should be able to eliminate the redundant analytic restrictions imposed on the displacement vector components u_i. We will not direct our investigations along this line and rather confine ourselves to a result of the same relative accuracy as that given in Chapter III. The approximate relation (3.9) will enable us then to obtain a number of relevant conclusions by rather elementary means.

Let us return to eq.(3.6). Since we can now approximately put

$$u_i \cong u_i^{(0)}, \tag{3.16}$$

we find from eq.(3.6)

$$u_i^{(0)} \stackrel{*}{=} \iint\limits_{(S)} X^j v^{(0)}_{(i)j}\, dS + \iiint\limits_{(V)} F^j v^{(0)}_{(i)j}\, dV - \iint\limits_{(S)} S^{(0)j}_{(i)} u_j dS \tag{3.17}$$

$$(i,\ j = 1,\ 2,\ 3).$$

where the superscripts (0) show that the auxiliary system of displacements corresponds to the load on the singular line, necessary to determine the coefficients $u_i^{(0)}$.

We shall now show that by varying the function $q(\alpha)$ we can, to within required accuracy, find the coefficients $u_i^{(1)}$, $u_i^{(2)}$ etc., without having to differentiate eqs.(3.9).

Indeed, for determining the coefficient $u_i^{(1)}$ with the necessary accuracy, it is sufficient to take the functions $q_2(\alpha)$ and $q_3(\alpha)$ as follows:

$$q_2(\alpha) = q_3(\alpha) = a_1\alpha + a_3\alpha^3 \tag{3.18}$$

and to determine the coefficients a_1 and a_3 from the conditions

$$n_1 = 1;\ n_3 = 0. \tag{3.19}$$

In this case, all coefficients of n_j with even subscripts j vanish. /304

Putting, as before, $\epsilon = 0.25$, we find the coefficients a_1 and a_3 from conditions (3.19). The functions $q_2(\alpha)$ and $q_3(\alpha)$ will be of the form

$$q_2(\alpha) = q_3(\alpha) = -\frac{81975}{18256604} h^{-3}\left(\alpha - \frac{13552}{5465}\frac{\alpha^3}{h^3}\right) \tag{3.20}$$

The coefficient $u_i^{(1)}$ will be determined by the following equation resulting from eq.(3.6):

$$u_i^{(1)} \doteq \iint\limits_{(S)} X^j v_{(i)j}^{(1)} dS + \iiint\limits_{(V)} F^j v_{(i)j}^{(1)} dV - \iint\limits_{(S)} S_{(i)}^{(1)j} u_j dS \tag{3.21}$$

$$(i,\ j = 1,\ 2,\ 3).$$

where the index (1) has a meaning similar to that of the superscript (0) in eq.(3.17).

Obviously, for an arbitrary coefficient $u_i^{(k)}$, after a suitable selection of the function $q(\alpha)$, we can set up the equation

$$u_i^{(k)} \doteq \iint\limits_{(S)} X^j v_{(i)j}^{(k)} dS + \iiint\limits_{(V)} F^j v_{(i)j}^{(k)} dV - \iint\limits_{(S)} S_{(i)}^{(k)j} u_j dS \tag{3.22}$$

$$(i,\ j = 1,\ 2,\ 3;\ k = 0\ \ 1,\ 2, \ldots,\ N).$$

It is not difficult to prove that eqs.(3.22) yield a new solution of the problem of reduction of the three-dimensional problems of the statics of an elastic body to the two-dimensional problems of shell theory. This solution does not require satisfaction of the condition that the components of the vectors of the forces acting on the shell be differentiable. We will explain the details of the new reduction method below.

Let us study the integrals on the right-hand sides of eqs.(3.17), (3.21), and (3.22). The integrals of the form $\iiint\limits_{(V)} F^j v_{(i)j}^{(k)} dV$ should be considered as prescribed functions of the coordinates of the point $M(x^i)$, i.e., as the points of intersection between the straight line bearing the additional load and the middle surface. The integrals $\iint\limits_{(S)}$ over the surface S of the shell can be represented in the form of sums of integrals over the boundary surfaces $S_{(\pm)}$ of the shell and its contour surface S_c. Since the load on the boundary surfaces of the shell is usually known, the integrals of the form $\iint\limits_{(\pm)} X^j v_{(i)j}^{(k)} dS$, where the sign $(\pm)$ has been introduced instead of the symbol $S_{(\pm)}$ to shorten the formulas, must be considered known functions. The integrals $\iint\limits_{(\pm)} S_{(i)}^{(k)j} u_j dS$ contain the covariant components of the required displacements on the boundary surfaces of the shell. Finally, the integrals over the contour surface S_c of the shell determine known and unknown functions. /305

The integral $\iint\limits_{(S_c)} X^j v_{(i)\,j}^{(k)}\, dS$ is a known function, if the forces acting on the contour surface are prescribed. Usually the forces acting on the part of the contour surface that is free from connectivity are known. On the parts of the contour surface with connectivity these forces are unknown, and the corresponding part of the integral $\iint\limits_{(S_c)} X^j v_{(i)\,j}^{(k)}\, dS$ will contain derivatives of the components of the required displacements of the points of the contour surface. Similarly, the integral $\iint\limits_{(S_c)} S_{(i)}^{(k)\,j} u_j dS$ is decomposed into two parts. That part of the integral taken over the region of the contour surface with prescribed displacement components is a known function. The other part of this integral contains components of the displacement sought.

On the basis of the properties of the auxiliary displacements considered in Sect.2, it can be asserted that the surface integrals will not be singular if the point $M(x^j)$ does not lie on the contour surface. If the point $M(x^j)$ does lie on the contour surface, then these integrals will be improper, but convergent. The volume integral will likewise be improper but convergent.

Bearing all the above in mind, let us now introduce the notation

$$\Phi_i^{(k)}(x^j) = \iiint\limits_{(V)} F^r v_{(i)\,r}^{(k)}\, dV + \iint\limits_{(\pm)} X^r v_{(i)\,r}^{(k)}\, dS + \iint\limits_{(\mathrm{I})} X^r v_{(i)\,r}^{(k)}\, dS -$$

$$- \iint\limits_{(\mathrm{II})} S_{(i)}^{(k)\,r} u_r' dS \qquad (i,\ r = 1,\ 2,\ 3;\ j = 1,\ 2;\ k = 0,\ 1,\ 2, \dots,\ N); \tag{3.23}$$

where $\iint\limits_{(\mathrm{I})}$ is the integral over that part of the contour surface with prescribed components of the forces of the basic system, and $\iint\limits_{(\mathrm{II})}$ is the integral over that part of the contour surface with the prescribed components of displacement of the basic system. Then, eqs.(3.22) take the form:

$$u_i^{(k)}(x^j) = \Phi_i^{(k)}(x^j) + \iint\limits_{(\mathrm{II})} X^r v_{(i)\,r}^{(k)}\, dS - \iint\limits_{(\mathrm{I})} S_{(i)}^{(k)\,r} u_r dS - \iint\limits_{(\pm)} S_{(i)}^{(k)\,r} u_r dS$$

$$(i,\ r = 1,\ 2,\ 3;\ j = 1,\ 2;\ k = 0,\ 1,\ 2, \dots,\ N), \tag{3.24}$$

and enable us to find the approximate expression for the displacement vector components at an arbitrary point of the shell.

Making use of eq.(3.9), after substituting the parameter α by the coordinate z, we obtain

$$u_i(x^j, z) = \sum_{k=0}^{N} z^k \Phi_i^{(k)}(x^j) + \iint_{(II)} X^r \sum_{k=0}^{N} z^k v_{(i)}^{(k)}{}_r\, dS -$$

$$- \iint_{(I)} u_r \sum_{k=0}^{N} z^k S_{(i)}^{(k)\,r}\, dS - \iint_{(\pm)} u_r \sum_{k=0}^{N} z^k S_{(i)}^{(k)\,r}\, dS$$

$$(i,\ r = 1,\ 2,\ 3;\ j = 1,\ 2). \tag{3.25}$$

The equalities (3.25) are approximate and noninvariant. The latter statement is connected with the fact that the scalar products in these equalities cannot be considered as absolute scalars. The properties of the integrals entering into eqs.(3.25) have already been discussed.

Thus, for an approximate determination of the field of displacements in the shell we must find the components of the displacements sought on the boundary surfaces of the shell and on part (I) of the contour surface, as well as the components of the basic system of forces on part (II) of the contour surface.

To solve the problem, we must set up equations in the above unknowns. Several versions for constructing the required systems of equations may be given. We confine ourselves in this Section to two versions.

1. Let us put z = ± h in eq.(3.25). Introduce the notation

$$u_i(x^j,\ h) = u_i^{(+)};\quad u_i(x^j,\ -h) = u_i^{(-)}. \tag{3.26}$$

Then, from eq.(3.25), we find the following system of integrodifferential equations:

$$u_i^{(+)} = \sum_{k=0}^{N} h^k \Phi_i^{(k)} + \iint_{(II)} X^r \sum_{k=0}^{N} h^k v_{(i)}^{(k)}{}_r\, dS - \iint_{(I)} u_r \sum_{k=0}^{N} h^k S_{(i)}^{(k)\,r}\, dS -$$

$$- \iint_{(+)} u_r^{(+)} \sum_{k=0}^{N} h^k S_{(i)}^{(k)\,r}\, dS - \iint_{(-)} u_r^{(-)} \sum_{k=0}^{N} h^k S_{(i)}^{(k)\,r}\, dS; \tag{3.27a}$$

$$u_i^{(-)} = \sum_{k=0}^{N} (-1)^k h^k \Phi_i^{(k)} + \iint_{(II)} X^r \sum_{k=0}^{N} (-1)^k h^k v_{(i)}^{(k)}{}_r\, dS -$$

$$-\iint\limits_{(1)} u_r \sum_{k=0}^{N} (-1)^k h^k S_{(i)}^{(k)\,r}\, dS -$$

$$-\iint\limits_{(+)} u_{(r)}^{(+)} \sum_{k=0}^{N} (-1)^k h^k S_{(i)}^{(k)\,r}\, dS - \iint\limits_{(-)} u_r^{(-)} \sum_{k=0}^{N} (-1)^k h^k S_{(i)}^{(k)\,r}\, dS$$

$$(i,\ r=1,\ 2,\ 3;\ j=1,\ 2). \qquad (3.27b)$$

Let us now assume that the system of auxiliary displacements is due to the action of the focusing load considered in Sect.2. Let the point $M(x^j,\ 0)$ on the middle surface lie outside the zone of width 2.5 Mh bordering its contour. Then, in eqs.(2.27a) - (3.27b) we may omit the integrals over the contour surface of the shell, and confine the integration region in the integrals over the boundary surfaces to the region lying within the circular cylinder of radius 2 Mh, with its axis coinciding with the straight line bearing the load of density $q(\alpha)$.

The system of equations (3.27a) - (3.27b) is now simplified and takes the form:

$$u_i^{(+)} = \sum_{k=0}^{N} h^k \Phi_i^{(k)} - \iint\limits_{(+)} u_r^{(+)} \sum_{k=0}^{N} h^k S_{(i)}^{(k)\,r}\, dS - \iint\limits_{(-)} u_r^{(-)} \sum_{k=0}^{N} h^k S_{(i)}^{(k)\,r}\, dS, \quad (3.28^a) \qquad (3.28a)$$

$$u_i^{(-)} = \sum_{k=0}^{N} (-1)^k h^k \Phi_i^{(k)} - \iint\limits_{(+)} u_r^{(+)} \sum_{k=0}^{N} (-1)^k h^k S_{(i)}^{(k)\,r}\, dS -$$

$$-\iint\limits_{(-)} u_r^{(-)} \sum_{k=0}^{N} (-1)^k h^k S_{(i)}^{(k)\,r}\, dS \qquad (3.28b)$$

$$(i,\ r=1,\ 2,\ 3;\ j=1,\ 2).$$

where the regions of integration (+) and (-) are bounded as noted above.

Let us assume that the point $M(x^1)$ lies in the zone of width 2 Mh bordering the contour of the middle surface. Then, in the integrals over the contour surfaces we must retain only the parts that correspond to the integration over the region enclosed within the circular cylinder having a radius 2 Mh and an axis coinciding with the straight line bearing a load of density $q(\alpha)$.

Thus the application of the focusing load permits us to restrict the integration region to relatively small regions with movable boundaries, varying /308 their position on displacement of the point $M(x^i)$ over the middle surface of the shell. In the following Section, we make use of equations with the integration regions restricted in this manner.

2. Let us now consider the second method of setting up the system of integrodifferential equations based on the application of eqs.(3.9) and (3.24). Assume that the right-hand side of eq.(3.9) contains an expression ensuring the approximate displacement of the vector u_i to the middle surface of the shell. We note that the elements of area on the boundary surfaces and on the middle surface are connected by the relations

$$dS_{(\pm)} = [1 \mp (k_1 + k_2) h + k_1 k_2 h^2]\, dS. \tag{3.29}$$

where $dS_{(\pm)}$ are elements of area of the boundary surfaces, dS is an element of area of the middle surface, and k_i are the principal curvatures of the middle surface.

Making use of the approximation equation (3.9), let us represent the components of the surface forces in the region (II) of the contour surface by the expansions

$$X^r \cong X^{(0)r} + zX^{(1)r} + \ldots + z^N X^{(N)r}. \tag{3.30}$$

Based on the approximate eqs.(3.9) and (3.30) and on eq.(3.29), we give eqs.(3.24) the following form:

$$\begin{aligned}
u_i^{(k)}(x^j) &= \Phi_i^{(k)}(x^j) + \int_{(\mathrm{II})}\int_{-h}^{+h} v_{(i)r}^{(k)} \sum_{p=0}^{N} z^p X^{(p)r}\, dz\, ds - \\
&- \int_{(\mathrm{I})}\int_{-h}^{+h} S_{(i)}^{(k)r} \sum_{p=0}^{N} z^p u_r^{(p)} dz\, ds - \iint_{(S)} \Big\{ \sum_{p=0}^{N} h^p u_r^{(p)} [1 - (k_1 + k_2) h + \\
&+ k_1 k_2 h^2] R_{(i)}^{(k)r} + \sum_{p=1}^{N} (-1)^p h^p u_r^{(p)} [1 + (k_1 + k_2) h + k_1 k_2 h^2] T_{(i)}^{(k)r} \Big\}\, dS.
\end{aligned} \tag{3.31}$$

where the integrals $\int_{(\mathrm{I})}$ and $\int_{(\mathrm{II})}$ are taken over those parts of the contour of the middle surfaces of the shell lying in regions (I) and (II) of the contour surface; the components $R_{(i)}^{(k)r}$ are equal to the vector components $S_{(i)}^{(k)r}$ on the

boundary surface z = +h, which vector has undergone parallel displacement to the middle surface; the components $T_{(i)}^{(k)\,r}$ are equal to the vector components $S_{(i)}^{(k)\,r}$ on the boundary surface z = -h, which vector has undergone parallel displacement to the middle surface*. /309

Let us introduce the notation:

$$v_{(ip)r}^{(k)} = \int_{-h}^{+h} z^p v_{(i)\,r}^{(k)} dz; \quad S_{(ip)}^{(k)\,r} = \int_{-h}^{+h} z^p S_{(i)}^{(k)\,r}\, dz; \tag{3.32a}$$

$$K_{(ip)}^{(k)\,r} = h^p \{[1-(k_1+k_2)h + k_1k_2h^2] R_{(i)}^{(k)\,r} + (-1)^p [1+(k_1+k_2)h + k_1k_2h^2] T_{(i)}^{(k)\,r}\}. \tag{3.32b}$$

Then, we obtain from eq.(3.31):

$$u_i^{(k)}(x^j) = \Phi_i^{(k)}(x^j) + \int_{(\mathrm{II})} v_{(ip)\,r}^{(k)} X^{(p)r}\, ds - \int_{(\mathrm{I})} S_{(ip)}^{(k)\,r} u_r^{(p)} ds - \iint_{(S)} K_{(ip)}^{(k)\,r} u_r^{(p)}\, dS \tag{3.33}$$

$$(i,\ r=1,\ 2,\ 3;\ j=1,\ 2;\ k,\ p=0,\ 1,\ 2,\ \ldots,\ N).$$

The signs of summation over (...p) are omitted.

Equations (3.33) must be regarded as a system of integrodifferential equations with unknown functions $u_i^{(k)}$ of a point of the middle surfaces of the shell. These equations are the integral analogs of the differential equations considered in Chapter III.

There is, however, a substantial difference between eqs.(3.33) and the equations of Chapter III. This difference lies in the fact that eqs.(3.33) do not contain the differentiation operation for the components of the vector forces acting on the shell. Consequently, these equations remain valid even in cases where concentrated forces are applied to the boundary surfaces of the shell. Obviously, this remark also applies to eqs.(3.27a) - (3.27b).

Now let us assume, as above, that the load on the singular line is focusing. Then, outside of the strip of width 2.5 Mh bordering the contour of the middle surface of the shell, eqs.(3.33) will be of the following form:

$$u_i^{(k)}(x^j) = \Phi_i^{(k)}(x^j) - \iint_{(\Sigma)} K_{(ip)}^{(k)\,r}\, u_r^{(p)} dS \tag{3.34}$$

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* We have taken advantage of the fact that the scalar product does not vary under parallel displacement, and this operation can be performed by separately displacing the cofactors (I, Sect.10.1).

$$(i,\ r=1,\ 2,\ 3;\ \ j=1,\ 2;\ \ k,\ p=0,\ 1,\ 2,\ \dots,\ N)$$

where the region (Σ) lies within a circular cylinder of radius 2.5 Mh and axis coinciding with the straight line bearing the load of density $q(\alpha)$.

If the point $M(x^i)$ of intersection between the middle surface and the load-carrying straight line lies in the strip of width 2.5 Mh bordering the contour of the middle surface, then the integration region (Σ) and the segments of arc of the contour to which the curvilinear integrals entering into eqs.(3.33) extend must be restricted to the inside of the circular cylinder having a radius 2.5 Mh and an axis coinciding with the straight line bearing the load $q(\alpha)$. /310

The equations of the form (3.34) are close in their mechanical meaning to the differential equations of Chapter III, since they describe the mechanical state of a small but finite part of the shell, while the differential equations describe the state of an element of the shell. It can be predicted that, by modifying the structure of the focusing load and passing to the limit, we will be able to find the differential equations of equilibrium from equations analogous to eqs.(3.34).

Several concluding statements are made below:

a) The preceding conclusions were based on the assumption of existence of a focusing load constructed by the method given in Sect.2. We shall not investigate this question further, since this involves several problems of mathematical analysis of the same nature as the classical problem of moments*. These problems cannot be discussed here and will be taken up in the second volume of this book. However, in Sect.9 we will give one of the other possible methods of constructing a focusing load, which involves no fundamental difficulties, especially questions of existence.

b) In setting up the integral equations (3.28a) - (3.28b) and (3.34), we restricted the region of integration to integrals containing the wanted functions. The same restriction of the integration regions can be carried out in the integrals containing the prescribed functions. These integrals enter into the right-hand side of eq.(3.23).

It is, however, necessary to determine first whether there are integrals over a region external to (Σ) and dominant relative to the integrals over the region (Σ). This case may occur, for example, if prescribed exterior forces are absent from the regions (Σ). Then, of course, the restriction of the integration region of integrals containing the prescribed functions to the region (Σ) cannot be applied.

c) Equations (3.28a) - (3.28b) and (3.34) are integral equations with variable integration limits, which are functions of the point $M(x^j)$ of intersection of the straight line bearing the load $q(\alpha)$ with the middle surface of the shell.

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* N.I.Akhiyezer, The Classical Problems of Moments. Fizmatgiz, 1961.

On displacement of the point $M(x^i)$ over the middle surface, the region (Σ) will cover the entire middle surface. In the boundary strip of width 2.5 Mh bordering the contour of the middle surface, the contour integrals enter into the equations, as already noted. /311

d) If the focusing load is constructed approximately, eqs.(3.28a) - (3.28b) and (3.34) should be considered as approximate. If the focusing load is constructed with sufficient accuracy, it will be possible to refine the value of the radius r_0 of the sphere which dissects the region (Σ) on the middle surface of the shell. For this purpose, and on the basis of elementary geometrical considerations, eq.(2.16b) must be changed to read

$$|\cos\varphi| \simeq \frac{r_0}{2R_{\min}}, \tag{d}$$

where R_{min} is the minimum radius of curvature of the middle surface in the vicinity of the point $M(x^i)$. Then, instead of the inequality $r_0 > 2.5|\alpha|$ we find

$$r_0 \geqslant |\alpha| \left(1 - \frac{|\alpha|}{R_{\min}}\right)^{-\frac{1}{2}} = |\alpha| \left(1 + \frac{1}{2}\frac{|\alpha|}{R_{\min}} + \dots\right), \tag{3.35}$$

and since $|\alpha|_{max} = Mh$, we obtain

$$r_0 \geqslant Mh\left(1 + \frac{1}{2}\frac{Mh}{R_{\min}} + \dots\right). \tag{3.36}$$

e) Application of focusing kernels, which is the foundation for constructing eqs.(3.28a) - (3.28b) and (3.34), leads to the conclusion that the boundary conditions have only a weak influence on the stress-strain state of the shell at points sufficiently distant from the contour of the middle surface.

Of course, the influence of the boundary conditions is not confined to the strip of width 2.5 Mh or to a narrower strip in accordance with the inequality (3.36), since when the point $M(x^i)$ goes beyond the boundary of this strip, the region (Σ) can include part of the bordering strip. For this reason, the influence of the boundary conditions is reflected on the stress-strain state of the entire shell. However, the structure of the above-derived equations permits the conclusion that the influence of the boundary conditions is weakened with increasing distance of the point $M(x^i)$ from the contour of the middle surface.

Section 4. Methods of Approximate Solution of a System of Integral Equations of Shell Theory

We recall that the approximate methods of solution of a system of integral equations is most often based on approximate substitution of these systems by

systems of algebraic equations. Use of focusing new kernels permits restricting the number of unknowns entering into the algebraic equations approximately equivalent to the integral equations. We will give one of the possible methods of setting up these equations.

Let us first consider the approximate representation of a certain function $\varphi(\xi, \eta)$ within the rectangle $M_1(x + a, y + b)$, $M_2(x - a, y + b)$, $M_3(x - a, y - b)$, $M_4(x + a, y - b)$, assuming that certain of its values $\varphi(x, y)$, $\varphi(x - a, y)$, $\varphi(x + a, y)$, $\varphi(x, y - b)$, $\varphi(x, y + b)$ are known. Then, interpolating the function $\varphi(\xi, \eta)$ by a paraboloid, we obtain /312

$$\begin{aligned}\varphi(\xi, \eta) \simeq \varphi(x, y) &+ \frac{[\varphi(x+a, y) - \varphi(x-a, y)](\xi - x)}{2a} + \\ &+ \frac{[\varphi(x, y+b) - \varphi(x, y-b)](\eta - y)}{2b} + \\ &+ \frac{[\varphi(x+a, y) + \varphi(x-a, y) - 2\varphi(x)](\xi - x)^2}{2a^2} + \\ &+ \frac{[\varphi(x, y+b) + \varphi(x, y-b) - 2\varphi(x)](\eta - y)^2}{2b^2}\end{aligned} \tag{4.1}$$

To shorten the formulas, let us introduce the following notation:

$$\begin{gathered}\varphi(x, y) = \varphi(0, 0),\ \varphi(x+a, y) = \varphi(1, 0),\ \varphi(x-a, y) = \varphi(-1, 0), \\ \varphi(x, y+b) = \varphi(0, 1),\ \varphi(x, y-b) = \varphi(0, -1).\end{gathered} \tag{4.2}$$

The absolute values of the coordinate increments will be denoted by h_j ($j = 1, 2$).

Let us return to the system of equations (3.34). The relative smallness of the integration region (Σ) permits us to represent the integrand functions by interpolation polynomials of the form of eq.(4.1), assuming the region (Σ) to be bounded by the rectangle $M_1M_2M_3M_4$ or to lie inside it. Using the notation (4.2), we obtain

$$\begin{aligned}u_i^{(k)}(0, 0) &+ u_r^{(p)}(0, 0)\iint_{(\Sigma)} K_{(ip)}^{(k)r}(0, \xi^j)\,dS + \\ &+ \frac{1}{2h_1}[u_r^{(p)}(1, 0) - u_r^{(p)}(-1, 0)]\iint_{(\Sigma)} K_{(ip)}^{(k)r}(0, \xi^j)(\xi^1 - x^1)\,dS + \\ &+ \frac{1}{2h_2}[u_r^{(p)}(0, 1) - u_r^{(p)}(0, -1)]\iint_{(\Sigma)} K_{(ip)}^{(k)r}(0, \xi^j)(\xi^2 - x^2)\,dS +\end{aligned}$$

$$+\frac{1}{2h_1^2}[u_r^{(p)}(1,\ 0)+u_r^{(p)}(-1,\ 0)-2u_r^{(p)}(0,\ 0)]\iint\limits_{(\Sigma)}K_{(ip)}^{(k)r}(0,\ \xi^j)\times$$

$$\times(\xi^1-x^1)^2\,dS+\frac{1}{2h_2^2}[u_r^{(p)}(0,\ 1)+u_r^{(p)}(0,-1)-2u_r^{(p)}(0,\ 0)]\times$$

$$\times\iint\limits_{(\Sigma)}K_{(ip)}^{(k)r}(0,\ \xi^j)(\xi^2-x^2)^2\,dS=\Phi_i^{(k)}(0,\ 0)$$

$$(i,\ r=1,\ 2,\ 3;\ j=1,\ 2;\ k,\ p=0,\ 1,\ 2,\ \ldots,\ N). \tag{4.3}$$

We recall that $N(\xi^j)$ is an arbitrary point of the region (Σ). The point /313 $M(x^j)$, as above, coincides with the intersection between the singular line bearing the auxiliary load and the middle surface of the shell.

Equations (4.3) constitute a system of linear algebraic equations. This system cannot be solved autonomously, since the number of unknowns is five times as great as the number of equations. However, by giving the coordinates x^j the increments $\pm h_j$, we obtain a system of equations general for the entire shell and containing the same number of wanted equations and unknowns.

We emphasize that the resultant system is more exact than that obtained by the usual methods, which are based on the substitution of partial derivatives by finite-difference ratios from the system of partial differential equations of shell theory, since integration "smoothes out" the errors introduced by the use of the interpolation formulas.

If we replace the derivatives by finite differences, neglecting small quantities of the first and higher orders, then the system of difference equations (4.3) is converted into a system of differential equations.

$$\frac{1}{2}\partial_j^2u_r^{(p)}\iint\limits_{(\Sigma)}K_{(ip)}^{(k)r}(x^s,\ \xi^s)(\xi^j-x^j)^2\,dS+\partial_ju_r^{(p)}\iint\limits_{(\Sigma)}K_{(ip)}^{(k)r}(x^s,\ \xi^s)\times$$

$$\times(\xi^j-x^j)\,dS+u_i^{(k)}+u_r^{(p)}\iint\limits_{(\Sigma)}K_{(ip)}^{(k)r}(x^s,\ \xi^s)\,dS=\Phi_i^{(k)}(x^s)$$

$$(i,\ r=1,\ 2,\ 3;\ j,\ s=1,\ 2;\ k,\ p=0,\ 1,\ 2,\ \ldots,\ N). \tag{4.4}$$

Equations (4.3) and (4.4) approximately describe the strained state of the shell, outside of the strip of width r_0 bordering the contour of the middle surface of the shell. Constructing equations suitable within this strip involves no difficulties. We shall not consider these equations here. The method of establishing them will be given later in the text.

Let us return to eqs. (4.3). It is not hard to convince ourselves that the first and second terms on the left-hand sides of these equations are dominant,

as results from the following considerations: Equations (3.34) can be represented in the form of

$$u_i^{(k)}(x^j)+u_r^{(p)}(x^j)\iint\limits_{(\Sigma)}K_{(ip)}^{(k)\,r}(x^j,\ \xi^j)\,dS+\iint\limits_{(\Sigma)}K_{(ip)}^{(k)\,r}(x^j,\ \xi^j)\times$$

$$\times[u_r^{(p)}(\xi^j)-u_r^{(p)}(x^j)]\,dS=\Phi_i^{(k)}(x^j)$$

$$(i,\ r=1,\ 2,\ 3;\ j=1,\ 2;\ k,\ p=0,\ 1,\ 2,\ \ldots,\ N). \tag{4.5}$$

A comparison of eqs.(4.3) and (4.5) shows that the first two terms on their left-hand sides coincide. The third term on the left-hand side of equation (4.5), after application of the interpolation formulas, is reduced to /314 the remaining terms on the left-hand side of eq.(4.3). For this reason, an evaluation of the third term on the left-hand side of eq.(4.5) is equivalent to an evaluation of the terms corresponding to it in eq.(4.3).

Assuming, according to the evaluations given in the monograph (Bibl.10), that

$$|\partial_j u_r^{(p)}|<\frac{2h}{L}, \tag{a}$$

where L is a characteristic dimension of the shell, we find for a thin shell at $2h : L = 0.01$; $M = 1.5$ and $L = R_{min}$:

$$\left|\iint\limits_{(\Sigma)}K_{(ip)}^{(k)\,r}(x^j,\ \xi^j)\,[u_r^{(p)}(\xi^j)-u_r^{(p)}(x^j)]\,dS\right|\ll 0{,}05\,(2h)^2\times$$

$$\times\sum_{(p,\,r)}\left|K_{(ip)}^{(k)\,r}(x^j,\ \xi_m^j)\right|, \tag{b}$$

where the point $N(\xi_m^j)$ lies between the point $M(x^j)$ and the boundary of the region (Σ). The evaluation (b) shows that the Gauss-Seidel iteration process is applicable to the system of equations (4.3)*. Retaining, for example, the first two terms on the left-hand side of eqs.(4.3) or (4.5), we obtain the system of equations of the first (initial) approximation:

$$u_i^{(k)}(x^j)+u_r^{(p)}(x^j)\iint\limits_{(\Sigma)}K_{(ip)}^{(k)\,r}(x^j,\ \xi^j)\,dS=\Phi_i^{(k)}(x^j)$$

$$(i,\ r=1,\ 2,\ 3;\ j=1,\ 2;\ k,\ p=0,\ 1,\ 2,\ \ldots,\ N). \tag{4.6}$$

* See, for instance, M.J.Salvadori, Numerical Methods in Engineering, IL,1955.

Equations (4.6) can evidently also be obtained from eqs.(3.34) by application of probability methods, for example of the Monte Carlo method *. The possibility of applying this method is based on the relative smallness of the region (Σ). In this case, the most correct choice of the value of the required function $u_r^{(p)}(\xi^j)$ under the integration sign is the choice of its value at the point $M(x^j)$ at the center of the region (Σ); we again obtain eqs.(4.6). The general analysis of the solvability of these equations is difficult, and we shall not give it here. We recall only that the Gauss method assumes the arbitrary choice of the initial values of the nondominating unknowns, permitting us, in the case of complications, to introduce into the right-hand side of eqs.(4.6) additional small terms, attributing arbitrary initial values to the rejected unknowns. The further cause of the Gauss-Seidel process is well known, and we shall not discuss it further. /315

The solutions of the system of algebraic equations (4.6) permit finding the initial approximate values of the required functions $u_i^{(k)}(x^j)$. Their dependence on the boundary conditions is reflected in the composition of the functions $\Phi_i^{(k)}(x^j)$, which contain all the assigned elements of the boundary conditions.

The solution of the system (4.6), consisting of $3(N + 1)$ equations, may be written out, using the well-known algebraic formulas

$$u_r^{(p)}(x^j) = \frac{\Delta_r^{(p)}(x^j)}{\Delta} \quad (r = 1,\ 2,\ 3;\ p = 0,\ 1,\ 2, \ldots, N). \tag{4.7}$$

This notation permits us, in a more easily visualized form than the tensor equation (4.6), to demonstrate the structure of the system of equations to be solved. The determinant of the system of equations (4.6) is of the form:

$$\begin{array}{ccccc}
u_1^{(0)} & u_2^{(0)} & u_3^{(0)} & & u_3^{(N)} \\
\end{array}$$

$$\Delta = \begin{vmatrix}
1+\iint\limits_{(\Sigma)} K_{(10)}^{(0)\,1}\,dS & \iint\limits_{(\Sigma)} K_{(10)}^{(0)\,2}\,dS & \iint\limits_{(\Sigma)} K_{(10)}^{(0)\,3}\,dS & \cdots\cdots & \cdots\iint\limits_{(\Sigma)} K_{(1N)}^{(0)\,3}\,dS \\
\iint\limits_{(\Sigma)} K_{(20)}^{(0)\,1}\,dS & 1+\iint\limits_{(\Sigma)} K_{(20)}^{(0)\,2}\,dS & \iint\limits_{(\Sigma)} K_{(20)}^{(0)\,3}\,dS & \cdots\cdots & \cdots\iint\limits_{(\Sigma)} K_{(2N)}^{(0)\,3}\,dS \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\iint\limits_{(\Sigma)} K_{(10)}^{(1)\,1}\,dS & \iint\limits_{(\Sigma)} K_{(10)}^{(1)\,2}\,dS & \iint\limits_{(\Sigma)} K_{(10)}^{(1)\,3}\,dS & 1+\iint\limits_{(\Sigma)} K_{(11)}^{(1)\,1}\,dS & \cdots\iint\limits_{(\Sigma)} K_{(1N)}^{(1)\,3}\,dS \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\iint\limits_{(\Sigma)} K_{(30)}^{(N)\,1}\,dS & \iint\limits_{(\Sigma)} K_{(30)}^{(N)\,2}\,dS & \iint\limits_{(\Sigma)} K_{(30)}^{(N)\,3}\,dS & \iint\limits_{(\Sigma)} K_{(31)}^{(N)\,1}\,dS & \cdots 1+\iint\limits_{(\Sigma)} K_{(3N)}^{(N)\,3}\,dS
\end{vmatrix} \tag{4.8}$$

* For the Monte Carlo method see, for instance, "Modern Mathematics for Engineers, E.F.Bekkenbakh, Editor, State Publishing House for International Literature, 1958, pp.275-287.

Each column of the determinant Δ corresponds to an unknown function $u_r^{(p)}$. /316
This is denoted by the notation above the horizontal line at the top of the determinant. To each value p, from 0 to N there correspond three columns with values of r equal to 1, 2, 3. The rows of the determinant correspond to the indices (k) and (i). To each value of k from 0 to N there correspond three rows with values of i equal to 1, 2, 3. This establishes the rule for setting up the determinants $\Delta_r^{(p)}$. We have

$$\begin{array}{cc} & \begin{array}{ccccc} u_1^{(0)} & \quad u_2^{(0)} & u_3^{(0)} & \quad u_r^{(p)} & \quad u_3^{(N)} \end{array} \\ \Delta_r^{(p)} = & \begin{vmatrix} 1+\iint\limits_{(\Sigma)} K_{(10)}^{(0)1}\,dS & \dots & \dots & \dots\Phi_1^{(0)}\dots & \iint\limits_{(\Sigma)} K_{1N}^{(0)3}\,dS \\ \dots\dots\dots & \dots & \dots & \dots\Phi_2^{(0)}\dots & \dots\dots \\ \dots\dots\dots & \dots & \dots & \dots\Phi_3^{(0)}\dots & \dots\dots \\ \dots\dots\dots & \dots & \dots & \dots\dots\dots & \dots\dots \\ \dots\dots\dots & \dots & \dots & \dots\Phi_3^{(N)}\dots & 1+\iint\limits_{(\Sigma)} K_{(3N)}^{(N)3}\,dS \end{vmatrix} \end{array} \qquad (4.9)$$

Of course, at high values of N, even this simplified solution involves a degree of computing work which, in labor involved, does not correspond to the requirements of relative accuracy adopted in the shell theory. As will be clear from Chapter III, we have $N \leq 3$. If N equals three, then the number of equations in the system (4.6) will be twelve. In this case, we must not use the "exact" methods of calculating the determinants $\Delta_r^{(p)}$ and Δ, but must have recourse to approximate methods. Further, as already noted, we must find the second approximation by substituting the required functions found from the first approximation into the left-hand sides of eqs.(4.3).

Equations (4.6) are applicable in the same region in which eqs.(3.34) are applicable, i.e., outside the strip determined by the inequality (3.36). If the point $M(x^i)$ is inside the strip bordering the contour of the middle surface of the shell, then the right-hand side of the equations will have the curvilinear integrals $\int\limits_{(\mathrm{I})}$ and $\int\limits_{(\mathrm{II})}$, which will extend over those parts of the contour lying inside a circular cylinder of radius r_0 with its axis coinciding with the straight line bearing the auxiliary load.

The task of investigating the stress-strain state of the shell inside this strip is highly complex. Here we have a substantially three-dimensional distribution of strains and stresses, so that a reduction of the three-dimensional problem of elasticity theory to the two-dimensional problem of shell theory /317
will undoubtedly distort reality, even if we construct an "exact" solution of the boundary problem of the shell theory. Evidently, the known solutions of the boundary problems of the shell theory, satisfying the classical boundary condi-

tions, describe only approximately the actual state of the shell close to its edges, although this approximation, as shown by observation and experiment, is sufficient for practical purposes. For this reason, there is not much use in trying to find the exact solutions of the boundary problems of shell theory if such solutions require a large expenditure of man-hours and money. We should instead attempt to construct an approximate solution, with an accuracy satisfying practical means.

We will not seek an exact solution in the border strip, since the entire method considered here is largely approximate. We will confine ourselves to the approximate solution, relying on the finite-difference method.

Let the arc of the contour have no corner points. Let us consider the family of normals to this arc. If the arc of the contour belongs to the segment (II), then the displacements are prescribed on it and on the corresponding part of the contour surface. At all points of the normal to the arc (II) of the contour on the boundary and beyond the border strip of width r_0, the displacement vector components are approximately known. By using linear interpolation we will then be able to construct, in first approximation, the displacement field within the zone and then to find the derivatives of the displacement components from the vectors of the local coordinate basis, belonging to the contour surface; by Hooke's law we will further be able to determine the components of the stress vector on part (II) of the contour surface. Then, by applying equations of the form of eq.(3.33), extending over the region (Σ), we introduce corrections into the displacement field determined in first approximation.

Continuation of this process is theoretically possible; its convergence is evidently ensured by the smoothing influence of integration.

The case of a region (Σ) lying inside the strip bordering the contour of the middle surface, and enclosing part of the contour (I), with the stress vector components prescribed on the contour surface, is somewhat more complicated than the last case. Here, the initial approximation by the displacement vector components on the contour surface must be determined by extrapolation in terms of the required values of the displacement vector components inside the zone and the values of these components outside the zone; we must set up modified equations (3.33) to determine the field of displacements inside the zone, since extrapolation introduces the required displacement vector components of an internal point of the zone into the integral $\int\limits_{(I)}$. We are unable to discuss /318

these questions in detail here, or to investigate the displacement fields in the neighborhood of a corner point of the contour.

Section 5. The Integrodifferential and Integral Equations of the Dynamics of Shells

In order to obtain the equations of motion from the equations of equilibrium, it is sufficient to include in the body forces the inertia forces

$$-\rho\frac{\partial^2 u^r}{\partial t^2}.$$

Equation (3.24) then takes the following form:

$$u_i^{(k)}(x^j, t) = \Phi_i^{(k)}(x^j, t) + \iint\limits_{(\mathrm{II})} X^r v_{(i)r}^{(k)} dS - \iint\limits_{(\mathrm{I})} S_{(i)}^{(k)r} u_r dS -$$

$$-\iint\limits_{(\pm)} S_{(i)}^{(k)r} u_r dS - \iiint\limits_{(V)} \rho v_{(i)r}^{(k)} \frac{\partial^2 u^r}{\partial t^2} dV$$

$$(i,\ r = 1,\ 2,\ 3;\ j = 1,\ 2;\ k = 0,\ 1,\ 2, \ldots, N). \tag{5.1}$$

We recall that an element of volume reads

$$dV = [1 - (k_1 + k_2)z + k_1 k_2 z^2]\, dz\, dS, \tag{5.2}$$

where dS is an element of area of the middle surface of the shell.

In the expressions of the inertia forces, let us pass to the covariant components, changing the arrangement of the indices in the corresponding scalar product entering into eq.(5.1). Finally, let us again make use of the approximate equation (3.9)

$$u_r \cong \sum_{p=0}^{N} z^p u_r^{(p)}. \tag{a}$$

Introduce now the notation

$$\Gamma_{(ip)}^{(k)r} = \int\limits_{-h}^{+h} \rho z^p v_{(i)}^{(k)r} [1 - (k_1 + k_2) z + k_1 k_2 z^2]\, dz. \tag{5.3}$$

Then, as a result of transformations analogous to those considered in deriving the system (3.33), we find

$$u_i^{(k)}(x^j, t) = \Phi_i^{(k)}(x^j, t) + \int\limits_{(\mathrm{II})} v_{(ip)r}^{(k)} X^{(p)r} ds - \int\limits_{(\mathrm{I})} S_{(ip)}^{(k)r} u_r^{(p)} ds -$$

$$-\iint_{(S)} K^{(k)\,r}_{(ip)}\, u^{(p)}_r dS - \iint_{(S)} V^{(k)\,r}_{(ip)} \frac{\partial^2 u^{(p)}_r}{\partial t^2} dS$$

$$(i,\ r=1,\ 2,\ 3;\ j=1,\ 2;\ k,\ p=0,\ 1,\ 2,\ \ldots,\ N). \qquad (5.4)$$

Equations (5.4) are to be regarded as a system of integrodifferential equations of the dynamics of shells in the unknown functions $u_i^{(k)}(t, x^j)$ where the point $M(x^j)$ belongs to the middle surface of the shell. /319

The statements made in discussing eqs.(3.33) are likewise applicable to eqs.(5.4).

Let us now consider two conditions of vibratory motion of the shell which are encountered in the solution of concrete problems.

1. Stationary Oscillatory Process

If the process is stationary, we may put

$$\Phi^{(k)}_i(t,\ x^j) = \Phi^{(k)}_{i0}(x^j) + \sum_{(\omega)} \Phi^{(k)}_{i\omega}(x^j)\cos(\omega t + \varepsilon_\omega)$$

$$(i=1,\ 2,\ 3;\ k=0,\ 1,\ \ldots,\ N). \qquad (5.5)$$

Similarly,

$$u^{(p)}_r(t,\ x^j) = u^{(p)}_{r0}(x^j) + \sum_{(\omega)} u^{(p)}_{r\omega}(x^j)\cos(\omega t + \varepsilon_\omega)$$

$$(r=1,\ 2,\ 3;\ p=0,\ 1,\ 2,\ \ldots,\ N). \qquad (5.6)$$

The sequence of frequencies ω may be either finite or infinite, but we will assume that this sequence is discrete. Substituting eqs.(5.5) and (5.6) into eqs.(5.4), and equating to zero the coefficients of $\cos(\omega t + \varepsilon_\omega)$, we find

$$u^{(k)}_{i0}(x^j) = \Phi^{(k)}_{i0}(x^j) + \int_{(\mathrm{II})} v^{(k)}_{(ip)\,r} X^{(p)\,r}_0\, ds - \int_{(\mathrm{I})} S^{(k)\,r}_{(ip)}\, u^{(p)}_{r0} ds -$$

$$- \iint_{(S)} K^{(k)\,r}_{(ip)}\, u^{(p)}_{r0} dS, \qquad (5.7)$$

$$u^{(k)}_{i\omega}(x^j) = \Phi^{(k)}_{i\omega}(x^j) + \int_{(\mathrm{II})} v^{(k)}_{(ip)\,r} X^{(p)\,r}_{\omega}\, ds - \int_{(\mathrm{I})} S^{(k)\,r}_{(ip)}\, u^{(p)}_{r\omega} ds -$$

$$- \iint_{(S)} [K_{(ip)}^{(k)r} - \omega^2 V_{(ip)}^{(k)r}] u_{r\omega}^{(p)} dS$$

$$(i, r = 1, 2, 3;\ j = 1, 2;\ k, p = 0, 1, 2, \ldots, N). \qquad (5.8)$$

where ω runs through a discrete series of values according to eqs.(5.5) - (5.6).

The system of equations (5.7) does not essentially differ from eqs.(3.33), and we shall therefore not discuss it. The system (5.8) contains the parameter ω. We will not further analyze the conditions of solvability of eqs.(5.8). Rather, we recall that, since the linear system of integrodifferential equations can be approximately replaced by a system of linear algebraic equations, it follows that (if, for some values of the parameter ω, there exists a non- /320 trivial solution of the homogeneous equations, i.e., equations with the free terms $\Phi_i^{(k)}$ equal to zero) the system of inhomogeneous equations (5.8) has no solution for these values of the parameter ω. These cases which are cases of resonance will require separate investigation.

2. Nonstationary Oscillatory Process

In transient regimes of loading of a shell, it may be impossible to represent the functions $\Phi_i^{(k)}(t, x^j)$ by equations of the form of eq.(5.5). In this case we must turn to the Laplace-Carson transformation of the system of integrodifferential equations (5.4).

It is well known that the representation of the functions f(t) according to Laplace and Carson is expressed by the following formula*:

$$\overset{*}{f}(p) = p \int_0^\infty e^{-pt} f(t)\, dt. \qquad (b)$$

Applying the Laplace-Carson transform to the second time derivative of the function f(t), we find

$$p \int_0^\infty e^{-pt} f''(t)\, dt = p^2 \overset{*}{f}(p) - p^2 f(0) - pf'(0). \qquad (c)$$

Thus the representation of the second derivative f"(t) is expressed in terms of the representation of the function f(t) and its value, and also its first time derivative at the initial time t = 0.

Applying the Laplace-Carson transform to the system of equations (5.4) and

* Cf., for instance, A.I.Lur'ye, Operational Calculus, Gostekhizdat, 1950.

using eqs.(b) and (c), we obtain the following system of operational integro-differential equations:

$$\overset{*}{u}{}_i^{(k)}(x^j, p) = \overset{*}{\Phi}{}_i^{(k)}(x^j, p) + p^2 \iint\limits_{(S)} V_{(iq)}^{(k)r} u_r^{(q)}(\xi^j, 0)\, dS +$$

$$+ p \iint\limits_{(S)} V_{(iq)}^{(k)r} u_r^{(q)}(\xi^j, 0)\, dS + \int\limits_{(\text{II})} v_{(iq)r}^{(k)} \overset{*}{X}{}^{(q)r} ds - \int\limits_{(\text{I})} S_{(iq)}^{(k)r} \overset{*}{u}{}_r^{(q)} dS -$$

$$- \iint\limits_{(S)} [K_{(iq)}^{(k)r} + p^2 V_{(iq)}^{(k)r}]\, \overset{*}{u}{}_r^{(q)}\, dS \tag{5.9}$$

$$(i, r = 1, 2, 3;\ j = 1, 2;\ k, q = 0, 1, 2, \ldots, N).$$

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The first three terms on the right-hand sides of eqs.(5.9) are prescribed functions. The initial conditions which must be satisfied by the wanted functions enter into these terms.

After determining the representations of the quantities $u_i^{(k)}$ from eqs.(5.9) we must find their originals, applying in the general case the Riemann-Mellin formula:

$$f(t) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} e^{pt} \frac{\overset{*}{f}(p)}{p}\, dp. \tag{d}$$

In special cases, Tables can be referred to that correlate the elementary and higher transcendental functions with their transforms*.

The described method was applied by G.Ye.Kazantseva in solving problems of vibration of round plates (G.Ye.Kazantseva, On the Vibrations of Thin Circular Plates. Thesis, Kiev Polytechnic Institute, 1956).

Section 6. Local Systems of Integrodifferential Equations of the Dynamics of Shells with Focusing Kernels and their Approximate Solution

Here, we will briefly discuss the approximate methods of solution of the equations derived in Sect.5, based on the focusing properties of systems of auxiliary loads and the related systems of auxiliary displacements of stresses.

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* Cf., for instance, V.A.Ditkin and A.P.Prudnikov, Integral Transformations and Calculus of Operations. Fizmatgiz, 1961.

As in the last Section, the equations of stationary and nonstationary oscillatory processes will be considered separately.

1. Stationary Oscillatory Processes.
The Frequency Spectrum

On the basis of the properties of equations with focusing kernels, the system (5.7) - (5.8), outside the strip of width r_0 bordering the contour of the middle surface, can be simplified and represented in the form:

$$u_{i0}^{(k)} = \Phi_{i0}^{(k)} - \iint\limits_{(\Sigma)} K_{(ip)}^{(k)\,r} u_{r0}^{(p)}\, dS, \tag{6.1a}$$

$$u_{i\omega}^{(k)} = \Phi_{i\omega}^{(k)} - \iint\limits_{(\Sigma)} [K_{(ip)}^{(k)\,r} - \omega^2 V_{(ip)}^{(k)\,r}]\, u_{r\omega}^{(p)}\, dS$$

$$(i,\ r=1,\ 2,\ 3;\ k,\ p=0,\ 1,\ 2,\ \ldots,\ N). \tag{6.1b}$$

The approximate solution of eqs.(6.1a) - (6.1b) can be performed as indicated in our discussion on the static equations, provided that the frequency of the free oscillations is not ω, i.e., that resonance is absent. /322

In connection with this question, let us consider the approximate determination of the frequency spectrum by use of the approximate solution (6.1b).

First, we note that the approximate equation (6.1b) and the more exact equations (5.8) of the shell theory permit an approximate determination only of a portion of the frequency spectrum of the three-dimensional problem of elasticity theory. The dimensions of the frequency regions accessible for determinations based on the equations of shell theory depend primarily on the number N, i.e., on the number of terms in the polynomials of z by which we approximated the displacement vector components. Further restriction of the frequency region depends on the approximate methods used in solving the integrodifferential equations of the shell theory.

To obtain an idea as to the effect exerted on the frequency equation when replacing the approximate equation (6.1b) with focusing kernels by equations with kernels that smoothly vary at any variation of the mutual position of the points $M(x^j)$ and $N(\xi^j)$ on the middle surface, let us have recourse to the classical argument based on the approximate replacement of eqs.(6.1b) by a system of algebraic equations.

Let us imagine, on the middle surface of the shell, an orthogonal coordinate net where the sides of the meshes are shorter than r_0, with r_0 equal to $Mh\left(1 + \frac{1}{2}\,\frac{Mh}{r_{min}} + \ldots\right)$ according to the condition (3.36). Then, the region (Σ) will include a finite number of nodes of the coordinate net. If n_1 is the

number of nodes of the coordinate net in the region (Σ), and n_0 is the number of nodes on the entire middle surface (S), then we obtain the approximate equation

$$n_1 : n_0 \approx (\Sigma) : (S). \tag{6.2}$$

We will consider the values of $u_i^{(p)}$ at the nodes of the coordinate net as unknowns. Then, placing the point $M(x^j)$ at the nodes of the net and approximately representing the integrals in eq.(6.1b) by finite sums under application of the formulas of mechanical quadrature, we replace the system of equations (6.1b) by systems of algebraic equations corresponding to fixed positions of the point M. This procedure is applicable both to equations with focusing kernels and to equations with kernels that do not possess focusing properties.

If we compare the determinant of the system of algebraic equations obtained from the integrodifferential equations having kernels without focusing properties with the determinants of the system resulting from eqs.(6.1b), we shall see that the latter has most of its elements equal to zero, since most of the nodes of the net, as can be seen from eqs.(6.2), are excluded from the region (Σ). /323

The determinant of the system with focusing kernels includes elements that depend on the boundary conditions. This fact is due to the influence of the boundary condition on the frequency spectrum. However, evidently there exists some part of the frequency spectrum that depends but weakly on the boundary conditions. We will offer suggestions that seem to confirm this conclusion. It must be emphasized that we consider the arguments developed below as being merely heuristic.

Let us perform, on the system (6.1b), simplifications analogous to those used in studying the static system (3.34). We find the system of algebraic equations analogous to the system (4.6):

$$u_{i\omega}^{(k)}(x^j) + u_{r\omega}^{(p)}(x^j) \iint\limits_{(\Sigma)} [K_{(ip)}^{(k)\,r} - \omega^2 V_{(ip)}^{(k)\,r}]\, dS = \Phi_{i\omega}^{(k)}$$

$$(i,\ r = 1,\ 2,\ 3;\ k,\ p = 0,\ 1,\ 2, \ldots,\ N). \tag{6.3}$$

The determinant of the system of algebraic equations (6.3) can be obtained from eq.(4.8) on replacing the kernel $K_{(ip)}^{(k)\,r}$ by the difference

$$K_{(ip)}^{(k)\,r} - \omega^2 V_{(ip)}^{(k)\,r}.$$

When the point $M(x^j)$ is superposed on the fixed nodes of the coordinate net, and when the integrals entering into the determinant Δ of the system of algebraic equations (6.3) are replaced by finite sums resulting from the formulas of mechanical quadrature, the determinant Δ will have numerical elements instead of functional ones. Equating the determinant Δ to zero, we find the

characteristic values of the parameter ω^2, the frequencies of the free vibrations of the shell.

The determinant Δ, as shown in Sect.4, is of the order $3(N + 1)$. If the number of nodes of the coordinate net on the middle surface S is n_0, then the total number of unknowns will be $3(N + 1)n_0$; this number is equal to the degree of the complete equation of frequencies with respect to ω^2. The roots of the complete equation of frequencies,i.e., of the equation including the coefficients of the unknowns at all nodes of the coordinate net, must include roots for which the determinant Δ, derived for the system of equation (6.3), approximately vanishes. In the opposite case, the homogeneous system of equations corresponding to eq.(6.3) would be only trivial solutions for these roots, which in turn would mean that the solutions of the system (6.3) had deviations arbitrarily great in absolute value from the solutions of the original system constructed without the above justified simplifications, a case which would contradict all of our earlier conclusions.

But then, to obtain approximately a portion of the frequency spectrum, /324 it is enough to equate the determinant Δ of the system of algebraic equation (6.3) to zero, placing the point $M(x^j)$ at one of the nodes of the coordinate net. Thus, we find the approximate values of the $3(N + 1)$ roots of the frewuency equation. Placing the point $M(x^j)$ at all the nodes of the net, we find all the $3(N + 1)n_0$ values of ω^2. This shows that the use of focusing kernels enables us approximately to represent the frequency equation as a product of n_0 factors.

This again leads to the concept of the existence of two groups of frequencies. The frequencies of one group are weakly connected with the boundary conditions. These are frequencies approximately determined by the equations that are obtained by equating the determinants of the system (6.3) to zero. The second group of frequencies depends substantially on the boundary conditions. These frequencies are obtained from the determinants of systems analogous to system (6.3) but set up under the assumption that the point $M(x^j)$ belongs to the strip of width r_0 bordering the contour of the middle surface. This subdivision of frequencies into two groups should become more distinct for thinner shells. Such phenomena are obviously connected with the dynamic boundary effects mentioned in A.Love's book*. An indirect confirmation of the correctness of these conclusions is the weak dependence of the critical values of the load on the boundary conditions, which is mentioned in works on the theory of stability of shells, in the presence of large regions on the middle surface sufficiently remote from its contour (Bibl.4, 10). The latter conclusions, however, require additional research.

2. Nonstationary Processes

In the case of equations with focusing kernels, eqs.(5.9) can be simplified if the point $M(x^j)$ lies outside the strip bordering the contour of the middle surface, as mentioned above. We find

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* A.Love, Mathematical Theory of Elasticity, ONTI, 1935, p.575.

$$\overset{*}{u}{}_i^{(k)} = \overset{*}{\Phi}{}_i^{(k)} + p^2 \iint\limits_{(\Sigma)} V_{(iq)}^{(k)\,r}\, u_r^{(q)}(\xi^j,\ 0)\, dS + p \iint\limits_{(\Sigma)} V_{(iq)}^{(k)\,r}\, \dot{u}_r^{(q)}(\xi^j,\ 0)\, dS -$$

$$- \iint\limits_{(\Sigma)} [K_{(iq)}^{(k)\,r} + p^2 V_{(iq)}^{(k)\,r}]\, \overset{*}{u}{}_r^{(q)}\, dS \tag{6.4}$$

$$(i,\ r = 1,\ 2,\ 3;\ j = 1,\ 2;\ k,\ q = 0,\ 1,\ 2, \ldots,\ N).$$

The simplifications made in setting up the equations of the statics of shells yield a system of algebraic equations analogous to system (4.3) and approximately replacing the system (6.4): /325

$$\overset{*}{u}{}_i^{(k)}(0,\ 0;\ p) + \overset{*}{u}{}_r^{(q)}(0,\ 0;\ p) \iint\limits_{(\Sigma)} [K_{(iq)}^{(k)\,r}(0,\ \xi^j) + p^2 V_{(iq)}^{(k)\,r}(0,\ \xi^j)]\, dS +$$

$$+ \frac{1}{2h_1}[\overset{*}{u}{}_r^{(q)}(1,\ 0;\ p) - \overset{*}{u}{}_r^{(q)}(-1,\ 0;\ p)] \iint\limits_{(\Sigma)} [K_{(iq)}^{(k)\,r}(0,\ \xi^j) +$$

$$+ p^2 V_{(iq)}^{(k)\,r}(0,\ \xi^j)]\,(\xi^1 - x^1)\, dS + \frac{1}{2h_2}[\overset{*}{u}{}_r^{(q)}(0, 1; p) - \overset{*}{u}{}_r^{(q)}(0,\ -1;\ p)] \times$$

$$\times \iint\limits_{(\Sigma)} [K_{(iq)}^{(k)\,r}(0,\ \xi^j) + p^2 V_{(iq)}^{(k)\,r}(0,\ \xi^j)]\,(\xi^2 - x^2)\, dS +$$

$$+ \frac{1}{2h_1^2}[\overset{*}{u}{}_r^{(q)}(1,\ 0;\ p) + \overset{*}{u}{}_r^{(q)}(-1,\ 0;\ p) - 2\overset{*}{u}{}_r^{(q)}(0,\ 0;\ p)] \times$$

$$\times \iint\limits_{(\Sigma)} [K_{(iq)}^{(k)\,r}(0,\ \xi^j) + p^2 V_{(iq)}^{(k)\,r}(0,\ \xi^j)]\,(\xi^1 - x^1)^2\, dS +$$

$$+ \frac{1}{2h_2^2}[\overset{*}{u}{}_r^{(q)}(0,\ 1;\ p) + \overset{*}{u}{}_r^{(q)}(0,\ -1;\ p) - 2\overset{*}{u}{}_r^{(q)}(0,\ 0;\ p)] \times$$

$$\times \iint\limits_{(\Sigma)} [K_{(iq)}^{(k)\,r}(0,\ \xi^j) + p^2 V_{(iq)}^{(k)\,r}(0,\ \xi^j)]\,(\xi^2 - x^2)^2\, dS = \overset{*}{\Phi}{}_i^{(k)}(0,\ 0;\ p) +$$

$$+ p^2 \iint\limits_{(\Sigma)} V_{(iq)}^{(k)\,r}(0,\ \xi^j)\, u_r^{(q)}(\xi^j,\ 0)\, dS + p \iint\limits_{(\Sigma)} V_{(iq)}^{(k)\,r}\, \dot{u}_r^{(q)}(\xi^j,\ 0)\, dS \tag{6.5a}$$

$$(i,\ r = 1,\ 2,\ 3;\ j = 1,\ 2;\ k,\ q = 0,\ 1,\ 2, \ldots,\ N).$$

The meaning of the notation introduced here is given in constructing the system of equations (4.3). Again basing our calculation on the scheme of the

Gauss-Seidel iteration process, we obtain the following system of equations of the first (initial) approximation:

$$\overset{*}{u}_i^{(k)}(x^j, p) + \overset{*}{u}_r^{(q)}(x^j, p) \iint\limits_{(\Sigma)} [K_{(iq)}^{(k)\,r}(0, \xi^j) + p^2 V_{(iq)}^{(k)\,r}(0, \xi^j)]\, dS =$$

$$= \overset{*}{\Phi}_i^{(k)}(x^j, p) + p^2 \iint\limits_{(\Sigma)} V_{(iq)}^{(k)\,r}\, u_r^{(q)}(\xi^j, 0)\, dS +$$

$$+ p \iint\limits_{(\Sigma)} V_{(iq)}^{(k)\,r}\, \dot{u}_r^{(q)}(\xi^j, 0)\, dS \qquad (6.5b)$$

$$(i, r = 1, 2, 3;\ j = 1, 2;\ k, q = 0, 1, 2, \ldots, N).$$

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Solving this system according to eqs.(6.7) on replacing the kernels $K_{(iq)}^{(k)\,r}$ by their sums $K_{(iq)}^{(k)\,r} + p^2 V_{(iq)}^{(k)\,r}$, we find the approximate expressions of the required representations $u_r^{(q)}$ in the form of integral rational functions of the parameter p, and then continue the process of iteration. The elementary considerations based on eqs.(4.7) and an evaluation of the degree of p in their numerators and denominators show that, from the resultant transforms, the originals can be found, i.e., these transforms permit inversion.

In the strip bordering the contour of the middle surface, we must use the approximate methods given in Sect.4. These methods again reduce the problem to the solution of a system of algebraic equations containing, as unknowns, the transforms of the functions sought. We will not further develop this method but return to the discrete-continuous method considered in Chapter IV.

Section 7. Application of the Discrete-Continuum Method

In Chapter IV we considered an approximate method for studying the dynamics of shells, based on replacing the shell by a discrete-continuous system. The meaning of the concept of a discrete-continuous system has been given in Sect.13, Chapter IV.

We recall that, to set up the equations of motion and the equations of connectivity that had to be satisfied by the generalized coordinates of the discrete-continuous system, we used a method of reducing the three-dimensional problems of the theory of elasticity to the two-dimensional problems of the theory of shells, based on expansion of the wanted functions in Maclaurin tensor series in ascending powers of the coordinate z.

This method has the shortcoming noted in Chapter III. The complications encountered in the analytic composition of the connectivity equations resulting from the boundary conditions are very substantial. We recall that these equations, in the general case, contained the second time derivatives (as well as time derivatives of orders higher than the second) of the generalized coordinates; this precludes the use of the apparatus of classical dynamics in con-

structing the equations of motion, although in special cases this apparatus is applicable.

In the preceding Sections of this Chapter we considered a new method of reduction, based on the theorem of work and reciprocity and the use of focusing auxiliary loads. This reduction method permits elimination of the above complications and to map out a more effective version of the discrete-continuous method.

We shall start out from the system of integrodifferential equations (5.4) with focusing kernels, still restricting the integration region to the region (Σ). Then, the system of integrodifferential equations (5.4), outside the strip of width r_0 bordering the contour of the middle surface, can be given the following form: /327

$$u_i^{(k)} = \Phi_i^{(k)} - \iint\limits_{(\Sigma)} K_{(ip)}^{(k)\,r} u_r^{(p)} dS - \iint\limits_{(\Sigma)} V_{(ip)}^{(k)\,r} \frac{\partial^2 u_r^{(p)}}{\partial t^2} dS \tag{7.1}$$

$$(i,\ r=1,\ 2,\ 3;\ k,\ p=0,\ 1,\ 2,\ \dots,\ N).$$

Let us make further use of the interpolation formula (4.1). We introduce the notation:

$$\begin{aligned}
2\Delta_1 u_r^{(p)} &= u_r^{(p)}(1,\ 0;\ t) - u_r^{(p)}(-1,\ 0;\ t);\\
2\Delta_2 u_r^{(p)} &= u_r^{(p)}(0,\ 1;\ t) - u_r^{(p)}(0,\ -1;\ t);\\
2\Delta_1^2 u_r^{(p)} &= u_r^{(p)}(1,\ 0;\ t) + u_r^{(p)}(-1,\ 0;\ t) - 2u_r^{(p)}(0,\ 0;\ t);\\
2\Delta_2^2 u_r^{(p)} &= u_r^{(p)}(0,\ 1;\ t) + u_r^{(p)}(0,\ -1;\ t) - 2u_r^{(p)}(0,\ 0;\ t).
\end{aligned} \tag{7.2}$$

Then, eqs.(7.1) will yield the system of equations analogous to the system (4.3):

$$\frac{\partial^2 u_r^{(p)}(0,\ 0;\ t)}{\partial t^2} \iint\limits_{(\Sigma)} V_{(ip)}^{(k)\,r}(0,\ \xi^j)\, dS + \frac{1}{h_1}\frac{\partial^2 \Delta_1 u_r^{(p)}}{\partial t^2} \iint\limits_{(\Sigma)} V_{(ip)}^{(k)\,r}(0,\ \xi^j) \times$$

$$\times (\xi^1 - x^1)\, dS + \frac{1}{h_2}\frac{\partial^2 \Delta_2 u_r^{(p)}}{\partial t^2} \iint\limits_{(\Sigma)} V_{(ip)}^{(k)\,r}(0,\ \xi^j)(\xi^2 - x^2)\, dS +$$

$$+ \frac{1}{h_1^2}\frac{\partial^2 \Delta_1^2 u_r^{(p)}}{\partial t^2} \iint\limits_{(\Sigma)} V_{(ip)}^{(k)\,r}(0,\ \xi^j)(\xi^1 - x^1)^2\, dS + \frac{1}{h_2^2}\frac{\partial^2 \Delta_2^2 u_r^{(p)}}{\partial t^2} \times$$

$$\times \iint\limits_{(\Sigma)} V_{(ip)}^{(k)\,r}(0,\ \xi^j)(\xi^2 - x^2)^2\, dS + u_i^{(k)}(0,\ 0;\ t) + u_r^{(p)}(0,\ 0;\ t) \times$$

$$\times \iint\limits_{(\Sigma)} K_{(ip)}^{(k)\,r}(0,\ \xi^j)\,dS + \frac{1}{h_1}\Delta_1 u_r^{(p)} \iint\limits_{(\Sigma)} K_{(ip)}^{(k)\,r}(0,\ \xi^j)(\xi^1 - x^1)\,dS +$$

$$+ \frac{1}{h_2}\Delta_2 u_r^{(p)} \iint\limits_{(\Sigma)} K_{(ip)}^{(k)\,r}(0,\ \xi^j)(\xi^2 - x^2)\,dS + \frac{1}{h_1^2}\Delta_1^2 u_r^{(p)} \times$$

$$\times \iint\limits_{(\Sigma)} K_{(ip)}^{(k)\,r}(0,\ \xi^j)(\xi^1 - x^1)^2\,dS + \frac{1}{h_2^2}\Delta_2^2 u_r^{(p)} \iint\limits_{(\Sigma)} K_{(ip)}^{(k)\,r}(0,\ \xi^j) \times$$

$$\times (\xi^2 - x^2)^2\,dS = \Phi_i^{(k)}(0,\ 0;\ t) \tag{7.3a}$$

$$(i,\ r = 1,\ 2,\ 3;\ j = 1,\ 2;\ k,\ p = 0,\ 1,\ 2,\ \ldots,\ N).$$

On the basis of the estimates given in Sect.4, we find from the system /328 (7.3a) the system of equations of the first (initial) approximation.

$$\frac{\partial^2 u_r^{(p)}(0,\ 0;\ t)}{\partial t^2} \iint\limits_{(\Sigma)} V_{(ip)}^{(k)\,r}(0,\ \xi^j)\,dS + u_i^{(k)}(0,\ 0;\ t) +$$

$$+ u_r^{(p)}(0,\ 0;\ t) \iint\limits_{(\Sigma)} K_{(ip)}^{(k)\,r}(0,\ \xi^j)\,dS = \Phi_i^{(k)}(0,\ 0;\ t)$$

$$(i,\ r = 1,\ 2,\ 3;\ j = 1,\ 2;\ k,\ p = 0,\ 1,\ 2,\ \ldots,\ N). \tag{7.3b}$$

The system of equations (7.3b) is analogous to the system of algebraic equations (4.6) of the statics of shells and contains derivatives only with respect to the time t.

Let us replace the triangulation net introduced in Sect.14 of Chapter IV by an orthogonal coordinate net corresponding to the interpolation formula (4.1). We shall superpose the point $M(x^j)$ with the nodes of the net and consider the quantities $u_r^{(p)}$ at the nodes as generalized coordinates. We assume that the meshes of the net considerably exceed the region (Σ). Then, in the region (Σ) there can be only one node of the net. Since the coordinates of the nodes are fixed, the integrals over the region (Σ) entering into eqs.(7.3b) will be functions of the number of nodes of the net. Bearing in mind eqs.(5.3), we introduce the notation:

$$m_{(ip)}^{(k)r}(n) = \iint\limits_{(\Sigma)} V_{(ip)}^{(k)r}(x_n^q,\ \xi^q)\,dS;$$

$$M_{(ip)}^{(k)rj}(n) = \iint\limits_{(\Sigma)} V_{(ip)}^{(k)r}(x_n^q,\ \xi^q)(\xi^j - x_n^j)\,dS;$$

$$I_{(ip)}^{(k)rj}(n)=\iint\limits_{(\Sigma)} V_{(ip)}^{(k)r}(x_n^q, \xi^q)(\xi^j-x_n^j)^2\,dS; \tag{7.4a}$$

$$c_{(ip)}^{(k)r}(n)=\iint\limits_{(\Sigma)} K_{(ip)}^{(k)r}(x_n^q, \xi^q)\,dS;$$

$$d_{(ip)}^{(k)rj}(n)=\iint\limits_{(\Sigma)} K_{(ip)}^{(k)r}(x_n^q, \xi^q)(\xi^j-x_n^j)\,dS;$$

$$e_{(ip)}^{(k)r}(n)=\iint\limits_{(\Sigma)} K_{(ip)}^{(k)r}(x_n^q, \xi^q)(\xi^j-x_n^j)^2\,dS; \tag{7.4b}$$

$$q_r^{(p)}(t, n)=u_r^{(p)}(x_n^j, t); \quad \Phi_i^{(k)}(t, n)=\Phi_i^{(k)}(x_n^j, t). \tag{7.4c}$$

where n is the number of the node of the net and the x_n^j are the coordinates of the node n

The system of equations (7.3a) takes the form /329

$$m_{(ip)}^{(k)r}(n)\frac{d^2q_r^{(p)}(t, n)}{dt^2}+\frac{1}{h_j}M_{(ip)}^{(k)rj}(n)\frac{d^2\Delta_j q_r^{(p)}(t, n)}{dt^2}+\frac{1}{h_j^2}I_{(ip)}^{(k)rj}\times$$

$$\times\frac{d^2\Delta_j^2 q_r^{(p)}(t, n)}{dt^2}+q_i^{(k)}(t, n)+c_{(ip)}^{(k)r}(n)\,q_r^{(p)}(t, n)+\frac{1}{h_j}d_{(ip)}^{(k)rj}(n)\times$$

$$\times\Delta_j q_r^{(p)}(t, n)+\frac{1}{h_j^2}e_{(ip)}^{(k)rj}(n)\Delta_j^2 q_r^{(p)}(t, n)=\Phi_i^{(k)}(t, n) \tag{7.5a}$$

(i, r = 1, 2, 3; j = 1, 2; k, p = 0, 1, 2, ..., N; the summation over j is performed according to the conventional rule).

The system of equations of the initial approximation is of the form:

$$m_{(ip)}^{(k)r}(n)\frac{d^2q_r^{(p)}(t, n)}{dt^2}+q_i^{(k)}(t, n)+c_{(ip)}^{(k)r}(n)\,q_r^{(p)}(t, n)=\Phi_i^{(k)}(t, n)$$

$$(i,\ r=1,\ 2,\ 3;\ k,\ p=0,\ 1,\ 2,\ldots,\ N). \tag{7.5b}$$

Equations (7.5a) and (7.5b) are set up for each node of the net that does not belong to the strip of width r_0 bordering the contour of the middle surface. The natural question arises as to the degree of approximation to the description of real motion of shell elements by the equations of the initial approximation (7.5b). This question is complex but is of practical importance, since a clear definition of the role of eqs.(7.5b) will decrease the amount of compu-

tational work necessary to solve the problem.

We shall confine ourselves to the following remark: Although eqs.(7.5b) are set up autonomously for each node of the net, which apparently is inconsistent with the interdependence between the motions of the nodes, this dependence is implicitly reflected in the composition of the coefficients of these equations and of their right-hand sides, which helped to reduce the error. Actually, the coefficients of eqs.(7.5a) - (7.5b) are expressed by integrals extending over a small but finite region (Σ) of the middle surface of the shell. Including, as they do, the elastic constants, the density of the shell, its thickness, and the principal curvatures of the middle surface, these coefficients reflect the basic properties of the shell as a continuous medium. They are links that connect the state of the shell at a certain point with its state in the region surrounding that point. This connectivity is the result of application of the Reciprocal Theorem, one of the fundamental theorems of the mechanics of elastic bodies.

In determining the first approximation from eqs.(7.5b), we find the following approximations from eqs.(7.5a), by substituting in them the second-degree terms of the solution of eqs.(7.5b). At these stages, the interrelation between the motions of adjacent nodes of the net is explicitly revealed.

In conclusion we note that, on application of equations that have ker- /330
nels without focusing properties, the integrals in the integrodifferential equations (5.4) would extend over the entire middle surface, and all the generalized coordinates would enter into them*.

Let us discuss the equations of motion of elements of a shell lying in a strip of width r_0, where r_0 is determined by eq.(3.36), bordering the contour of the middle surface of a shell. The greatest difference between these equations and the equations considered in Chapter IV (Sects.15 - 16) lies in the fact that components of the displacement and stress vectors, prescribed over the contour surface, are included in the prescribed functions Φ_i^k, which are here analogs of the generalized forces. This recalls the "automatic" inclusion of the equations of geometric connectivity in the system of Lagrange equations of the second kind on selection of the generalized coordinates corresponding to the conditions of a concrete problem of mechanics without introduction of redundant coordinates. Evidently, it is not a question here of a simple external similarity but of a complex internal relation, since the Reciprocal Theorem is connected with the principle of possible displacements**.

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* The above properties for equations with focusing kernels and the resultant simplifications of the system of equations of motion of shells were given by us in the Note "Approximate Methods of Investigating the Equilibrium and Vibrations of Shells as Discrete-Continuous Systems", in: Information Bulletin No.2 of the Scientific Council on "Scientific Principles of Strength and Plasticity", 1961.

** This interrelation was called to our attention by Yu.N.Shevchenko, Senior Scientist, Institute of Mechanics, Academy of Sciences UkrSSR.

Thus, the complications mentioned in Sect.15 of Chapter IV do not actually arise. It is true, these complications were due to the method of reduction used in that Section and did not result from the essence of the mechanical problem. We recall that the appearance of time derivatives in the equations of connectivity was due, in this case, to the successive elimination of derivatives of the form $\nabla_3 \ldots \nabla_3 u_1$ on the basis of the equations of motion in the Lamé form.

To set up the differential equations of motion of the nodes of the coordinate net in the strip bordering the contour of the middle surface, it is necessary, as pointed out in Sect.4, to use unilateral finite differences for expressing the unknown values of the components of displacement and stress on the contour surface in terms of their values inside the shell. The forms of these equations depend substantially on the form of the contour and the geometrical properties of the middle surface. Here, these equations are not considered.

The discrete-continuous method always allows us to obtain a system of ordinary differential equations, determinate in the sense that the number of equations equals the number of functions sought. This confirms the existence and uniqueness of the solution of the system of differential equations of motion of the shell, constructed by us.

We recall that the approximate replacement of the integrals by finite /331 sums, has been, ever since Euler's day, the classical method of investigating various questions of mathematical physics*. The requirements of mathematical rigor compel us to pass to the limit, from the wanted systems of differential equations with a finite number of functions to systems of integrodifferential equations. This passage to the limit must show that the solutions of the systems of differential equations obtained from the integrodifferential equations (5.4) by interpolation formulas, i.e., by the discrete-continuous method, coincide at the limit with the solutions of the initial system (5.4).

Of course, the solutions of the simplified systems of the form of equations (7.5a) - (7.5b) are only rather rough approximations to the exact solutions, and cannot be used to construct a sequence of functions converging to the exact solution of the system (5.4).

We have not investigated the above-mentioned process of convergence in detail since the technique of such investigation has been thoroughly studied so that its result is known in advance, although one would have to overcome considerable difficulties when investigating a method of mathematically describing the process adapted to the special features of the problem. Such investigations would be outside the scope of our study.

Section 8. Nonlinear Integrodifferential Equations of the Dynamics of Shells

In Sect.12 of Chapter II, we proved a theorem which we called the theorem of work and reciprocity in the nonlinear theory of elasticity. This theorem is

* See, for instance, R.Courant and D.Gilbert, Methods of Mathematical Physics, Vol.1 - 2, Gostekhizdat, 1933 - 1951.

applicable to an anisotropic medium with geometrical and physical nonlinearity.

The Reciprocal Theorem makes it possible to construct nonlinear integro-differential equations of the dynamics of shells.

We shall confine ourselves to consideration of a shell made of an isotropic material, and shall consider only the case of geometrical nonlinearity. It is natural to assume that finite deformations are developed in the shell under the action of the applied load and under certain conditions on its surface contour, as a result of its flexibility (Bibl.4). For this reason finite deformations characterize the basic state of the shell but not its auxiliary state.

As before, the auxiliary state will be constructed from the displacements /332 and stresses in a linearly deformed unbounded medium under the action of the load indicated in Sect.2. In this case, the shell, as already pointed out, must be regarded as part of an unbounded medium.

This argument eliminates the fundamental difficulty connected with the above procedure for application of the Reciprocal Theorem, since it is then no longer necessary to determine the particular solutions of the equations of the theory of elasticity, corresponding to the action of a concentrated force in a nonlinear deformable elastic medium.

Let us again consider (II, 12.10) as well as (II, 12.2a), (II, 12.2b), (II, 12.8) and (II, 12.9). We shall attempt to reduce these relations to expressions containing the basic system of required functions $u_r^{(p)}$ introduced in this Chapter. To obtain results suitable for concrete calculations, we shall introduce, into the nonlinear part of the components of the finite-deformation tensor D_{ik}, only those terms which have been considered dominant ever since the first studies made by T.Karman. These are the terms depending on the displacement vector components u_3 (Bibl.4, 10). Let us use the notation:

$$\Delta_{rs} = \nabla_r u^j \nabla_s u_j \simeq \nabla_r u^3 \nabla_s u_3 \simeq \nabla_r u_3 \nabla_s u_3 \quad (r,\ s = 1,\ 2,\ 3). \tag{8.1}$$

We make use here of the metric of the unstrained shell, in which, in accordance with Sect.2 of Chapter II, the covariant derivatives ∇_r and ∇_s are determined. We note that, in contrast to the modern theory of flexible plates and shells, we shall not put $(\nabla_3 u_3)^2 \approx 0$.

The covariant derivatives $\nabla_i u_3$ (i = 1, 2) do not enter into the system of functions $u_r^{(p)}$. Therefore, to accomplish the program of setting up equations in the unknown functions $u_r^{(p)}$, we shall use the approximate relation resulting from the equalities (III, 7.4a), supplementing it by the nonlinear factor

$$\nabla_i u_3 \simeq \nabla_i u_3^{(0)} \simeq \left[\frac{X_{(+)i} - X_{(-)i}}{2\mu} - u_i^{(1)} \right] (1 + u_3^{(1)})^{-1} \qquad (i = 1,\ 2). \tag{8.2}$$

where $X_{(+)i}$ and $X_{(-)i}$ are the stress vector components determined with respect to the forces acting on the deformed boundary surfaces. This results from the conditions (II, 8.2b), which also have a meaning for finite deformations.

From eqs.(8.2), the following approximate relation is obtained:

$$\Delta_{rs} \cong [Y_r Y_s - Y_r u_s^{(1)} - Y_s u_r^{(1)} + u_r^{(1)} u_s^{(1)}](1 + u_3^{(1)})^{-2} \qquad (r, s = 1, 2);$$

$$\Delta_{3s} = u_3^{(1)} \nabla_s u_3 = u_3^{(1)} [Y_s - u_s^{(1)}](1 + u_3^{(1)})^{-1} \tag{8.3a}$$

$$(s = 1, 2); \tag{8.3b}$$

$$\Delta_{33} = (u_3^{(1)})^2. \tag{8.3c}$$

We introduced the notation /33

$$Y_i = \frac{X_{(+)i} - X_{(-)i}}{2\mu} \qquad (i = 1, 2). \tag{8.4}$$

Under the above assumptions, the relation (II, 12.2b) can be represented as follows:

$$2T_{ik} = \lambda g_{ik} g^{rs} \Delta_{rs} + 2\mu \Delta_{ik} \cong \lambda g_{ik} \sum_{l=1}^{2} g^{ll} [Y_l - u_l^{(1)}]^2 (1 + u_3^{(1)})^{-2} +$$
$$+ \lambda (u_3^{(1)})^2 + 2\mu (Y_i Y_k - Y_k u_i^{(1)} - Y_i u_k^{(1)} + u_i^{(1)} u_k^{(1)})(1 + u_3^{(1)})^{-2}; \tag{8.5a}$$

$$T_{i3} = \mu u_3^{(1)} [Y_i - u_i^{(1)}](1 + u_3^{(1)})^{-1}; \quad T_{33} = \frac{1}{2} (u_3^{(1)})^2 \tag{8.5b}$$
$$(i, k = 1, 2).$$

We took account here of the orthogonality of the system of coordinates x^j on the undeformed middle surface.

Making use of eqs.(II, 12.2a) and (8.5a) - (8.5b), we find, for the basic system of forces acting on the shell and the resultant strains and stresses:

$$\sigma_{*ik} = \sigma_{ik} - \frac{1}{2} \lambda g_{ik} \sum_{l=1}^{2} g^{ll} [Y_l - u_l^{(1)}]^2 (1 + u_3^{(1)})^2 -$$
$$- \mu (Y_i Y_k - Y_k u_i^{(1)} - Y_i u_k^{(1)} + u_i^{(1)} u_k^{(1)})(1 + u_3^{(1)})^{-2} \tag{8.6a}$$
$$(i, k = 1, 2);$$

$$\sigma_{*j3} = \sigma_{j3} - \mu u_3^{(1)} [Y_j - u_j^{(1)}] (1 + u_3^{(1)})^{-1}; \quad \sigma_{*33} = \sigma_{33} - \frac{1}{2} (u_3^{(1)})^2$$

$$(j = 1, 2). \tag{8.6b}$$

Further, it follows from (II, 12.8) that

$$F_{ni} = \sigma_{*ik} n_0^k = \sigma_{ik} n_0^k - T_{ik} n_0^k. \tag{a}$$

The meaning of the second term on the right-hand side of eq.(a) will be clear from the foregoing. The first term is connected with the stress vector on the surface of the shell, as will be seen from (II, 8.13). To write eq.(a) in the expanded form, one must separately consider the stress vector on the boundary surfaces of the shell and on its contour surface. On the boundary surfaces, eqs.(II, 8.3) can be represented in the form

$$x^i = \xi^i \qquad (i = 1, 2); \; x^3 = \pm h; \tag{b}$$

and, on the contour surface,

$$x^i = x^i(\xi^1) \qquad (i = 1, 2): \; x^3 = \xi^2. \tag{c}$$

Making use of (II, 8.9b), we find on the boundary surfaces: /334

$$B_{33} = 1. \tag{d}$$

All the remaining quantities B_{rs} on the boundary surfaces vanish. On the contour surface

$$B_{11} = \left(\frac{dx^2}{d\xi^1}\right)^2; \quad B_{12} = B_{21} = -\frac{dx^1}{d\xi^1}\frac{dx^2}{d\xi^1}; \; B_{22} = \left(\frac{dx^1}{d\xi^1}\right)^2; \quad B_{r3} = 0$$

$$(r = 1, 2, 3). \tag{e}$$

Making use of (II, 6.5) and of the simplifying assumptions by T.Karman, we obtain on the boundary surfaces of the shell:

$$A^{rs,pq} B_{rs} \varepsilon_{pq} = A^{33,pq} \varepsilon_{pq} \cong A^{33,p3} \varepsilon_{p3} = -2\varepsilon_{33} = -2u_3^{(1)} \tag{f}$$

and, applying (II, 8.13), we find on the boundary surfaces of the shell:

$$\sigma_{ij} n_0^j \cong \pm \frac{X_{(\pm)i}}{1+u_3^{(1)}} = \pm X_{(\pm)i}(1 - u_3^{(1)} + \ldots) \quad (i=1,\ 2). \tag{8.7}$$

Noting that on the boundary surfaces of the shell, for $z = \pm h$,

$$n_0^j = 0 \quad (j=1,\ 2); \quad n_0^3 = \pm 1, \tag{8.8}$$

we find from eq.(a),

$$F_{ni}\Big|_{z=\pm h} = \pm \frac{X_{(\pm)i}}{1+u_3^{(1)}} - \mu[Y_i - u_i^{(1)}] \frac{u_3^{(1)}}{1+u_3^{(1)}}, \quad (i=1,\ 2); \tag{8.8a}$$

$$F_{n3}\Big|_{z=\pm h} = \pm \frac{X_{(\pm)3}}{1+u_3^{(1)}} - \frac{1}{2}(u_3^{(1)})^2. \tag{8.8b}$$

We note that the quantities $u_3^{(1)}$ are small by comparison with $u_i^{(1)}$. This permits a simplification in deriving, from the equations of motion, the equations of first approximation. Essentially eqs.(8.8a) - (8.8b) show that, under Karman's simplifying assumptions, the stress vector components on the boundary surfaces can be determined by the linear theory. If, according to Karman's evaluations, the order of $u_i^{(1)}$ is equal to the order of $\sqrt{\epsilon_{ik}}$ (i, k = 1, 2), and the order of $u_3^{(1)}$ is higher than the order of ϵ_{ik}, then all introduced nonlinear terms will be of the order $\sim \mu\epsilon_{ik}^{3/2}$ and $\mu\epsilon_{ik}^2$.

If terms with the factors ϵ_{pq} (p, q = 1, 2) are retained in eqs.(8.8a) to (8.8b), then the right-hand sides of eqs.(8.8a) - (8.8b) will have additional /335 terms of the order $\sim \mu\epsilon_{ik}^2$. Of course, if considerable accuracy is desired, one must retain in eqs.(8.8a) - (8.8b) first of all the terms of order $\sim \mu\epsilon_{ik}^{3/2}$.

On the contour surface,

$$A^{rs.\,pq} B_{rs} \varepsilon_{pq} \cong A^{rs.\,p3} B_{rs} \varepsilon_{p3} \cong A^{rs.\,p3} B_{rs} \left[Y_p - \frac{1}{2} u_3^{(1)} \nabla_p u_3 \right]. \tag{g}$$

Here, we dropped the factor $(1 + u_3^{(1)})^{-1}$. Making use of eqs.(8.2), we find

$$A^{rs.\,pq} B_{rs} \varepsilon_{pq} \cong A^{rs.\,p3} B_{rs} \left[Y_p \left(1 - \frac{1}{2} u_3^{(1)} \right) + \frac{1}{2} u_3^{(1)} u_p^{(1)} \right]. \tag{8.9}$$

Here again the term $Y_p u_3^{(1)}$ is of the order $\sim \epsilon_{ik}^2$, and this term should be neglected. The $u_3^{(1)} u_p^{(1)}$ is of the order $\sim \epsilon_{ik}^{3/2}$. Such terms can be retained in the construction of the second approximation.

On the contour surface,

$$n_0^j \neq 0 \quad (j=1,\ 2); \quad n_0^3 = 0. \tag{h}$$

Let us assume, to simplify the calculations, that there are no loads on the contours of the boundary surfaces. In that case, the components Y_p vanish. Again making use of (II, 8.13), and relation (a) of this Section, we obtain on the contour surface

$$F_{ni} \cong X_{ni}\left(1+\frac{A^{rs,\,p3}B_{rs}u_3^{(1)}u_p^{(1)}}{4B_{rs}g^{rs}}\right) - T_{i1}n_0^1 - T_{i2}n_0^2 \tag{8.10}$$

$$(i=1,\ 2,\ 3).$$

The components T_{ik} are determined from eqs.(8.5a) - (8.5b). It will be clear from eqs.(8.5a) that the right-hand side of eq.(8.10) contains the nonlinear terms $u_i^{(1)} u_k^{(1)}$ (i, k = 1, 2), of the order of $\sim \mu\epsilon_{ik}$. Thus, in contrast to the stress vector components on the boundary surfaces, the components of the vector on the contour surface are substantially nonlinear, even if we use the simplified Karman theory.

The calculations performed here show that the only sources of substantially nonlinear terms, which are of the relative order $\sim \mu\epsilon_{ik}$, in the stress tensor components, are represented by the components of the tensor T_{ik} (i, k = = 1, 2). This could have been foreseen. Indeed, the substantially nonlinear terms depend on the components of the antisymmetric tensor Ω_{rs} rather than on the tensor of small deformations ϵ_{rs}. This is confirmed, for instance, by (II, 9.2).

Hereafter, to simplify the calculations we shall omit terms of the relative order $\sim \mu\epsilon_{ik}^{3/2}$ and of higher orders. /336

Let us now consider the body forces Φ^k, determined from (II, 12.9). First, let us study all terms containing the quantities $P_{ik\cdot}^{\cdot\cdot j}$. In accordance with the relative accuracy adopted, the quantities $P_{ik\cdot}^{\cdot\cdot j}$ must be replaced by the components of the tensor $N_{ik\cdot}^{\cdot\cdot j}$ of rank two, determined by (II, 6.12b). On the basis of the above study we conclude that the sum

$$\Sigma = N_{ij\cdot}^{\cdot\cdot l}\,\sigma_*^{jk} + N_{ij\cdot}^{\cdot\cdot k}\,\sigma_*^{lj} \tag{i}$$

does not contain terms of the relative order $\sim \mu\epsilon_{ik}$ and should be omitted. Consequently,

$$\Phi^k \cong \rho F^k + \nabla_i T^{ik} - \rho \frac{\partial^2 u^k}{\partial t^2}$$
$$(i,\ k = 1,\ 2,\ 3). \qquad (8.11)$$

After completing the preliminary analysis, we make use of (II, 12.10), expressing the generalized Reciprocal Theorem. The above expression (II, 12.10), under the previous assumptions on the auxiliary system of forces, displacements, and stresses, again leads to an equation of the form of eq.(5.1), with several additional terms, including the integral

$$I = \iiint\limits_{(V)} v_{(i)r}^{(k)} \nabla_s T^{sr} dV. \qquad (8.12)$$

Making use of the Ostrogradskiy-Gauss formula, let us transform the integral I as follows:

$$\iiint\limits_{(V)} v_{(i)r}^{(k)} \nabla_s T^{sr} dV = \iiint\limits_{(V)} \nabla_s [v_{(i)r}^{(k)} T^{sr}]\, dV - \iiint\limits_{(V)} T^{sr} \nabla_s v_{(i)r}^{(k)} dV =$$
$$= \iint\limits_{(S)} v_{(i)r}^{(k)} T^{sr} n_s dS - \iiint\limits_{(V)} T^{sr} \nabla_s v_{(i)r}^{(k)} dV = \iint\limits_{(S)} v_{(i)}^{k(r)} T_{sr} n^s dS -$$
$$- \iiint\limits_{(V)} g^{sj} T_{jr} \nabla_s v_i^{(k)r} dV. \qquad (k)$$

From eqs.(a) and (k) of this Section it will be seen that the integrals $\iint\limits_{(S)}$ over the surface of the shell, containing the components of the tensor T_{ik}, are cancelled.

We neglect the nonlinear terms in the composition of the surface forces /337 depending on $u_3^{(1)}$ according to eqs.(8.8a) - (8.8b) and (8.10), since these terms are of an order of smallness higher than $\mu \varepsilon_{ik}^{3/2}$.

Consequently, at the degree of relative accuracy adopted by us, the only source of nonlinear terms in the integrodifferential equations of the theory of shells will be the integral

$$I_1 = - \iiint\limits_{(V)} g^{sj} T_{jr} \nabla_s v_{(i)}^{(k)r} dV \qquad (j,\ r = 1,\ 2). \qquad (8.13)$$

Making use of eq.(8.5a), and retaining in it the terms of the order $\sim \mu e_{ik}$, we obtain

$$I_1 = \iint\limits_{(S)} H^{(k)qq}_{(i)} u^{(1)}_q u^{(1)}_q dS + \iint\limits_{(S)} R^{(k)\,jr}_{(i)} u^{(1)}_j u^{(1)}_r dS. \tag{8.14}$$

where

$$H^{(k)qq}_{(i)} = -\frac{\lambda}{2}\int\limits_{-h}^{+h} g^{sj} g_{jr} g^{qq} \nabla_s v^{(k)r}_{(i)} [1-(k_1+k_2)z + k_1 k_2 z^2]\, dz =$$

$$= -\frac{\lambda}{2}\int\limits_{-h}^{+h} g^{qq} \nabla_r v^{(k)r}_{(i)} [1-(k_1+k_2)z + k_1 k_2 z^2]\, dz; \tag{8.15a}$$

$$R^{(k)jr}_{(i)} = -\mu\int\limits_{-h}^{+h} g^{sj} \nabla_s v^{(k)r}_{(i)} [1-(k_1+k\ )z + k_1 k_2 z^2]\, dz. \tag{8.15b}$$

The integral I_1 enters into the right-hand side of eq.(5.4) with a positive sign, yielding

$$u^{(k)}_{(i)}(x^j,\ t) = \Phi^{(k)}_i(x^j,\ t) + \int\limits_{(II)} v^{(k)}_{(ip)\,r} X^{(p)\,r} ds - \int\limits_{(I)} S^{(k)r}_{(ip)} u^{(p)}_r ds -$$

$$- \iint\limits_{(S)} R^{(k)jq}_{(i)} u^{(1)}_j u^{(1)}_q dS + \iint\limits_{(S)} H^{(k)qq}_{(i)} u^{(1)}_q u^{(1)}_q dS + \iint\limits_{(S)} K^{(k)r}_{(ip)} u^{(p)}_r dS -$$

$$- \iint\limits_{(S)} V^{(k)r}_{(ip)} \frac{\partial^2 u^{(p)}_r}{\partial t^2} dS \tag{8.16}$$

$$(i,\ r=1,\ 2,\ 3;\ j,\ q=1,\ 2;\ k,\ p=0,\ 1,\ 2,\ \dots,\ N).$$

Equations (8.16) form the required system of nonlinear integrodifferential equations of the theory of shells. This system occupies a place intermediate between the system of equations (5.4) of the linear theory and the general system of equations of "strong flexure". However, so far as we know, the latter system has never been set up or at least has not yet been seriously studied. /338

Let us return to the system (8.16). Let us assume, as above, that the auxiliary system of loads leads to the construction of focusing kernels of integrodifferential equations. Then, outside a strip of width $Mh\left(1 + \frac{1}{2}\frac{Mh}{R_{\min}} + \dots\right)$,

bordering the contour of the middle surface, the system (8.16) takes the following form:

$$u_i^{(k)}(x^j,\ t) = \Phi_i^{(k)}(x^j,\ t) - \iint\limits_{(\Sigma)} K_{(ip)}^{(k)r}\, u_r^{(p)} dS + \iint\limits_{(\Sigma)} [H_{(i)}^{(k)qq}\, u_q^{(1)} u_q^{(1)} +$$

$$+ R_{(i)}^{(k)jq}\, u_j^{(1)} u_q^{(1)}]\, dS - \iint\limits_{(\Sigma)} V_{(ip)}^{(k)r} \frac{\partial^2 u_r^{(p)}}{\partial t^2}\, dS \qquad (8.17)$$

$$(i,\ r = 1,\ 2,\ 3;\ j,\ q = 1,\ 2;\ k,\ p = 0,\ 1,\ 2,\ \ldots,\ N).$$

Applying interpolation formulas of the type of eq.(4.1), we replace the system of integrodifferential equations (8.17) by a system of nonlinear differential equations.

We shall only give the system of equations of the initial approximation, since the construction of a system analogous to eq.(7.5a) involves no fundamental difficulties. We find

$$\frac{\partial^2 u_r^{(p)}(0,\ 0;\ t)}{\partial t^2} \iint\limits_{(\Sigma)} V_{(ip)}^{(k)r}(0,\ \xi^j)\, dS + u_i^{(k)}(0,\ 0;\ t) + u_r^{(p)}(0,\ 0;\ t) \times$$

$$\times \iint\limits_{(\Sigma)} K_{(ip)}^{(k)r}(0,\ \xi^j)\, dS - u_q^{(1)}(0,\ 0;\ t)\, u_q^{(1)}(0,\ 0;\ t) \iint\limits_{(\Sigma)} H_{(i)}^{(k)qq}(0,\ \xi^j)\, dS -$$

$$- u_j^{(1)}(0,\ 0;\ t)\, u_q^{(1)}(0,\ 0;\ t) \iint\limits_{(\Sigma)} R_{(i)}^{(k)jq}(0,\ \xi^j)\, dS = \Phi_i^{(k)}(0,\ 0;\ t)$$

$$(i,\ r = 1,\ 2,\ 3;\ j,\ q = 1,\ 2;\ k,\ p = 0,\ 1,\ 2,\ \ldots,\ N). \qquad (8.18)$$

If the shell is in equilibrium, then the system of differential equations (8.18) is transformed into a system of nonlinear algebraic equations. The solution of the system (8.18), and of the more general and analogous system (7.5a), is, in the general case, a non-single-valued function. In order to select the required branch of the solution, one must investigate the gradual development of deformations of the shell from its initial undeformed state. The branch points or limit points will correspond to the critical values of the load.

One of the practical methods of solving the system of equations (8.17) is to replace it by a system of ordinary differential equations, using the dis- /339
crete-continuous method. Making use of a coordinate net on the middle surface, superposing the point $M(x^j)$ on the nodes of the net, and considering the quantities $u_r^{(p)}$ at the nodes as generalized coordinates, we shall obtain a system of ordinary differential equations analogous to the systems (7.5a) - (7.5b). We

recall that these systems could be constructed for sufficiently strong focusing properties of the kernels of the integrodifferential equations and for dimensions of the net such that, on superposition of the point $M(x^j)$ on one of the nodes in the region (Σ), there would be no neighboring nodes. In this case, as noted in Sect.7, the integrals over the region (Σ) entering into eqs.(8.18) will be functions of the numbers of the nodes of the net. Let us make use of the notation (7.4a) - (7.4c) and introduce the additional notation

$$h_{(i)}^{(k)ss}(n) = \iint\limits_{(\Sigma)} H_{(i)}^{(k)ss}(x_n^j, \xi^j)\, dS; \quad r_{(i)}^{(k)js}(n) = \iint\limits_{(\Sigma)} R_{(i)}^{(k)js}(x_n^j, \xi^j)\, dS. \tag{8.19}$$

Here, as in Sect.7, n is the number of a node of the net.

The system of equations (8.18) now takes the following form:

$$m_{(ip)}^{(k)r}(n)\frac{d^2 q_r^{(p)}(t, n)}{dt^2} + q_i^{(k)}(t, n) + c_{(ip)}^{(k)r}(n)\, q_r^{(p)}(t, n) -$$

$$- h_{(i)}^{(k)ss}(n)\, q_s^{(1)}(t, n)\, q_s^{(1)}(t, n) - r_{(i)}^{(k)js}(n)\, q_j^{(1)}(t, n)\, q_s^{(1)}(t, n) = \Phi_i^{(k)}(t, n)$$

$$(i, r = 1, 2, 3;\ j, s = 1, 2;\ k, p = 0, 1, 2, \ldots, N;\ n = 1, 2, \ldots, S). \tag{8.20}$$

The above statements on consideration of the system of equations (7.5b) can be applied, with certain additions, to the system of equations (8.20).

Since, under the simplifying assumptions adopted by us, the nonlinear terms disappear from the boundary conditions, the equations of motion in the zone bordering the contour of the middle surface will not differ substantially from eqs.(8.20). The method of approximate construction of the equations in the bordering zone indicated in Sect.7 remains valid also in this case.

Finally we note that the application of the method of equivalent linearization, analyzed in Chapter IV, makes it possible to replace the system (8.20) by the linear system of equations (7.5b), provided the elastic constants are properly substituted.

In the concluding Section of this Chapter we will draw generalizing conclusions on the proposed methods and give a general characterization of their significance for the mechanics of shells.

Section 9. On the Construction of Kernels of Integrodifferential Equations with Focusing Properties

The preceding portion of this Chapter was based on the possibility of constructing a system of auxiliary displacements, leading to kernels of integrodifferential equations with properties which, in accordance with C.Lanczos*, we

* See the passage cited in Section 2 of the book by C.Lanczos.

will call focusing. It will be clear from the content of Sects.2 and 3 that an ideal focus is evidently impossible, just as it is impossible in optical instruments. However, as we have shown in Sects.2 and 3, a focusing property sufficient to guarantee the required accuracy of the results can indeed be obtained.

However, the use of the theory of generalized functions opens new possibilities for the construction of kernels with ideal focusing properties, i.e., properties corresponding to the perturbations of an elastic medium embracing a strictly bounded region of space and vanishing beyond those bounds. These applications of the theory of generalized functions would go beyond the scope of the present study.

In Sects.2 and 3, the focusing action of the load was obtained by constructing the functions $q_1(\alpha)$ and $q_4(\alpha)$ according to eqs.(2.27) - (2.29). These functions determined the distribution of the singularities of the field of auxiliary displacements and stresses on the segments of the singular line going beyond the boundaries of the region bounded by the surface of the shell. As already noted, the exact solution of the problem of the required distribution of singularities can go beyond the limits of classical analysis. If we abandon the requirement of "exact focusing", then the methods of constructing the focusing load given in Sects.2 and 3 can be simplified.

Let us give up the determination of the functions $q_1(\alpha)$ and $q_4(\alpha)$. The field of displacements and stresses caused in the shell by action of the loads $q_2(\alpha)$ and $q_3(\alpha)$ will likewise possess weak focusing properties. To strengthen these properties, let us multiply the displacements caused by the loads $q_2(\alpha)$ and $q_3(\alpha)$ by a function of the point N, which is damped with sufficient rapidity with increasing distance of the point N from the singular line, and which, on the singular line, becomes unity.

An example of a factor strengthening the focusing action of the load is the function

$$F(r) = e^{-kf(r)} \tag{9.1}$$

where

$$r = \sqrt{\sum_{i=1}^{3} (y_i - \eta_i)^2}, \tag{9.2}$$

$k \gg 0$, while $f(r)$ is a monotonically increasing, everywhere differentiable, and continuous function satisfying the condition

$$f(0) = 0 \tag{9.3}$$

as well as the condition that the derivatives $\left.\frac{\partial f}{\partial y_i}\right|_{y_i=\eta_i}$ shall be bounded.

We recall that y_i and η_i are the rectangular Cartesian coordinates em- /341
ployed in Sect.2.

The factor of the form (9.1) introduced into the right-hand sides of equations (2.1a) - (2.1b) permits constructing the field of displacements satisfying the inhomogeneous equations of equilibrium of the elastic body caused by the fields of body forces in the unbounded elastic medium, approaching zero as r increases without limit. The point $M(\eta_i)$, if the condition (9.3) is satisfied and if the derivatives $\left.\frac{\partial f}{\partial y_i}\right|_{y_i=\eta_i}$ are bounded, will, as before, be the point of application of the unit concentrated force. The further constructions of the integrodifferential equations do not in principle differ from those considered above.

A positive coefficient k can always be chosen such that, on the boundary of the assigned region (Σ) and outside that region, the positive values of the components of the auxiliary displacements, stresses and body forces, shall not exceed prescribed small quantities. This will ensure the focusing properties of the kernels of the integrodifferential equations.

All above elementary conclusions require no detailed proof, since they result from the well-known analytic properties of the particular solutions of the equations of statics of an elastic medium, corresponding to the action of concentrated forces. They do, however, confirm the existence of integrodifferential equations of the theory of shells with focusing kernels, which are of fundamental significance in the statics and dynamics of shells.

Section 10. Integrodifferential Equations Defining Contiguous Solutions of the Boundary Problems of the Statics and Dynamics of Shells

Here we will briefly characterize another method of setting up and solving the integrodifferential and integral equations of the shell theory. This method was first given by us in 1946 (Bibl.23e) and was further developed later (Bibl.23f), (Bibl.23i). Several reports have been published in the meantime, in which this method is applied to various problems of the statics of shells*.

The problems considered by these methods are special cases of the general problem which can be defined as follows: The solution of a boundary problem of the statics or dynamics of shells is known. Required, to construct the solution of a different problem close to the first one by some criterion.

This problem was posed by us previously (Bibl.23e). Solutions close by some definite criterion we term contiguous or adjacent. In the above works, this problem was solved mainly in one version. The solution of a boundary /342
problem of the equilibrium of a plate, onto whose middle plane was mapped the middle surface of a shell, was assumed to be known. Then, a system of integrodifferential equations or, in particular, integral equations, was constructed

* We might mention the works (Bibl.17, 31a,b; Bibl.34a,b).

which yielded the solution of an adjacent boundary problem for the shell. An exception was another study (Bibl.31b), where the solution of the boundary problem of statics of a cylindrical shell was used for solving the boundary problem of statics of a cylindrical shell with modified boundary conditions.

The primary means for constructing the integrodifferential and integral equations of an adjacent problem was the application of the Reciprocal Theorem according to our previous studies (Bibl.23e) which we later developed in great detail (Bibl.23h). The essence of these considerations reduces to the following: It is usually asserted that the Reciprocal Theorem connects two systems of displacements and the corresponding forces or stresses in some elastic body (II, Sect.12). The interpretation of this theorem can be expanded by considering two systems of displacements in two different elastic bodies with mutually connected arithmetization of their interior points by systems of coordinates having the same relative dimensionality. Then the system of displacements of the second body may be attributed to the first body, by determining the external forces corresponding to these displacements from the equations of the theory of elasticity.

This treatment of the theorem of work and reciprocity has proved fruitful, yielding new approaches to the solution of problems, not only of the mechanics of shells but also of the statics and dynamics of one-dimensional and three-dimensional problems of the theory of elasticity and plasticity*. It also became possible to separate, from the displacements of points of the middle surface, the "plate terms" from the terms that depend on the curvature of the middle surface. The separation of the "plate terms" has helped to solve a number of problems, mainly on the equilibrium of cylindrical shells, although the method has been developed for shells of arbitrary form (Bibl.23g-i).

Limited space prevents us from examining this method in more detail; we have, therefore, found it more expedient to focus the reader's attention on the new methods of application of the Reciprocal Theorem, leading to the construction of equations with focusing kernels.

Several statements should be made in conclusion:

1. The described method can be further improved by introducing focusing factors of the form of eq.(9.1) into the system of auxiliary displacements. On approximate replacement of the system of integrodifferential equations of equilibrium of a shell by a system of algebraic equations, this will permit us to /343 decrease the number of unknowns in each equation.

2. The method of constructing equations that determine the adjacent solutions permits constructing a chain or continuity of solutions in which each solution results from the preceding solution. This, of course, requires a large amount of work.

3. The system of integral equations of equilibrium of the theory of shells,

* Cf., for example (Bibl.23c), as well as the paper by N.A.Kil'chevskiy and L.V.Lirsa, and the paper by R.A.Mikhaylenko in Izv. Kiev. Politekhn. Inst., Vol.31, 1961.

obtained by transformation of the integrodifferential equations, can be a system of Fredholm integral equations of the second kind. This system, however, may have no unique solution, if the conditions of Fredholm's third theorem are satisfied. An example of this case is given elsewhere (Bibl.23j).

Section 11. Concluding Remarks on the Integrodifferential and Integral Equations of the Statics and Dynamics of Shells

We are here giving general conclusions on the role played by the results of the investigations given in Chapter V for the general theory of shells. Here we must distinguish between the purely theoretical value and the applied value of the methods considered. Let us first characterize the theoretical value of the method of integrodifferential equations resulting from the Reciprocal Theorem.

We have shown that the application of the Reciprocal Theorem supplements the solution methods for the problem of reducing the three-dimensional problems of the theory of elasticity to the two-dimensional problems of the theory of shells by a substantially new method.

One of the advantages of this method, in our opinion, is that its application does not require that the components of the body and surface forces applied to the shell be differentiable, in contrast to the methods indicated at the beginning of Chapter III, which require satisfaction of the conditions that the vector components of the prescribed forces be differentiable. In (III, Sect.19) we note that the solution of the equations resulting from the general equation of dynamics likewise requires that the vector components of the prescribed forces be differentiable or that the theory of generalized functions be applied. These difficulties are eliminated when the methods given in this Chapter are applied. The smoothing influence of integration permits us, without analytic complications, to consider a shell loaded by concentrated forces.

In this connection, we must emphasize the fundamental difference between the methods considered in Sects.1 - 9 of this Chapter and the methods mentioned in Sect.10. This difference consists in the fact that the method in Sect.10 is not an independent method of reduction, but is based on application of the results of a preliminary reduction.

Several works have recently been published on new methods for setting up the integral equations of the statics and dynamics of shells. These works give methods that permit replacement of the system of differential equations of equilibrium or motion of an element of a shell, compatibly with the boundary conditions, by equivalent systems of integrodifferential or integral equations*. /344

These methods are related to the method given in Sect.10. It is clear that the results of the replacement of the system of differential equations by an equivalent system of integral equations will not eliminate the error introduced into the system of differential equations by the application of various

* Cf. A.A.Berezovskiy, Integrodifferential Equations of the Nonlinear Theory of Flat Thin Shells, Ukr.Matemat. Zhurnal, Vol.XII, 1960; and (Bibl.19).

simplifying assumptions, for example, the Kirchhoff-Love assumptions, V.Z.Vlasov's simplifying assumptions of the technical theory of shells, and others. For this reason, the cognitive value of these methods is lower than that of the general method considered in Sects.1 - 9, which autonomously solves the reduction problem.

The method considered in Sects.1 - 9 permits of further generalizations, based on the distribution of singularities of the displacements of the auxiliary problem, not on a line but in part of the volume of the shell. In this case, we can construct various generalized averages of the components of the displacement vector and the components of the strain and stress tensors, thus eliminating the complications connected with the preliminary approximate representations of the displacement vector components by eqs.(3.9).

We will discuss now the practical value of the above methods. The method based on the application of integrodifferential and integral equations with focusing kernels, as is obvious from the content of Sects.1 - 9, is an effective means for a numerical solution of the boundary problems of the shell theory. In exactly the same way, the methods given in Sect.10 can be used as the basis for a numerical solution of these boundary problems. It seems to us that the methods considered are more adaptable to the application of numerical computational methods than systems of differential equations, since the replacement of the integral by a finite sum introduces a smaller error than the replacement of derivatives by a ratio of finite differences. The inconvenience connected with the necessity of extending the sums, approximately replacing integrals, over all nodes of the coordinate net of the middle surface, which leads to equations with a large number of unknowns, is eliminated by the introduction of focusing kernels. Of course, all numerical methods require the use of computers.

The principal shortcoming of the study in Chapter V is the lack of calculated analytic expressions and Tables of focusing kernels. We have confined /345 ourselves to a description of the theoretical principles of the method, since construction of the analytic expressions for the kernels and the corresponding Tables is very tedious and would take a relatively long time. These data will be published in the next part of this study.

It is clear that a realistic study of focusing kernels might compel the introduction of particular corrections to the above-given methods of their construction, but there can be no doubt that this would not affect the fundamental principles of the method itself.

A shortcoming inherent in the entire study is the fact that we neglected the dissipative forces of various origin generated during the motion of the elements of the shell. We deliberately adopted this extensive simplification, since our object was to set forth the principles of the analytic theory of shells, and either the introduction or the neglecting of dissipative forces does not go to the foundation of the theory.

We recall that, in many papers on the dynamics of shells, the effect of the dissipative forces has likewise not been subjected to investigation. It was noted in these cases that the influence of the dissipative forces, as shown

by experiment, causes a rapid damping of free oscillations (Bibl.12). This inconsistency, which sometimes appears in the dynamics of shells, is eliminated in a number of more recent studies, including the Bolotin monograph (Bibl.2c).

The method of integral and integrodifferential equations in the theory of shells has not received general recognition. As A.I.Lur'ye so vividly puts it, its "competitiveness" still requires confirmation.

We assume that the development of the theory of equations with focusing kernels and examples of their application to special problems will yield convincing confirmation of the power of this new method. It is particularly important to characterize the field of special problems in which this method has obvious advantages over the methods of classical shell theory. This field should be that of particularly complex problems. We recall that the methods of analytic mechanics have long been applied to problems of particular difficulty, whose solution could not be directly obtained from the laws of Newton and the general theorems of dynamics. D.Leach writes that the application of analytic mechanics to the solution of simple problems is as inefficient as is the use of an airplane to cross a street*. This statement is fully applicable to the analytic mechanics of shells.

The field of problems for which it is obviously expedient to apply the new methods will gradually become outlined**.

* D.Leach, Classical Mechanics. IL, Moscow, 1961.

** The first step in this direction has been taken. Cf.,interalia,(Bibl.17,31b)

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